CONVOLUTIONAL CODES FOR CPM USING THE MEMORY OF THE MODULATION PROCESS

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ABSTRACT
The modulation process of continuous phase modulation (CPM) is interpreted as a redundant encoder with memory and a signal table. It is shown that by a fusion of this encoder with a preceding convolutional encoder, the total number of delay elements and by this the number of states, which have to be traced at the receiver, can be reduced. At first a mapping rule is used between the codewords and the signal, for which the memory of the modulation process can be expressed by a linear, nonrecursive circuit, and therefore leads to an obvious form of the fusion of both encoders. Thereafter the method is extended to mapping rules with a recursive memory structure and arbitrary phase pulses.

I. INTRODUCTION
In several papers (e.g.,[2]-[5]) convolutional codes are given, with which the reliability of digital modulation with continuous phase is increased. The spectral efficiency of these schemes is conserved by transmitting the codes' redundancy by means of an increased signal space. In this paper it is shown that the inherent memory of the modulation process can be used by the convolutional encoder. This results in a decreased number of encoder-modulator states, which have to be traced at the receiver or an increased coding gain respectively. For that purpose the modulation process is divided into a redundant encoder with memory and a signal table (cf.[6]). This point of view leads to many others to the usual mapping rules between the codewords and the signal. On the other hand the knowledge of the internal structure of the memory of the modulation process makes plain redundant encoder-modulator states in a fusion of both encoders. At first the method is explained only for CPM-signals ([1]). In section VI the results are generalized to CPM with other baseband phase pulses. For several examples optimum codes and coding gains are given.

II. SIGNAL DESCRIPTION AND DEFINITIONS
The same block diagram (Fig. 1) and signal description of coded CPM are used as in ref. [2], [4]. Each time interval a convolutional encoder [10] with \( v \) binary delay elements and rate \( r = k/n \) generates a codeword \( X = (x_1, x_2, \ldots, x_{k+1}) \) with binary symbols \( x_i \) from \( k \) binary source symbols \( \underline{x} = (u_1, u_2, \ldots, u_k) \). In a mapper, which in contrast to [2][5], may contain memory itself, \( M \) -ary phase transition factors \( a = (a_1, a_2, \ldots, a_{(N-1)}) \) are coordinated to the codewords \( X \). The CPM-modulator produces the transmitter signal:

\[
s(t, \alpha) = \sqrt{\frac{E}{T}} \cos(\psi(t, \alpha))
\]

(1)

\( E \): Signal energy per symbol interval \( T \)

The phase function \( \psi(t, \alpha) \) is given by:

\[
\psi(t, \alpha) = 2\pi f_c t + \phi_0 + 2\pi h \sum_{i=0}^{L-1} a_i q(t - iT)
\]

(2)

\( f_c \): Carrier frequency

\( h \): Modulation index

\( q(t) \): Phase pulse with the restriction:

\[
q(t) = \begin{cases} 0, & t < 0 \\ 1/2, & t \geq LT 
\end{cases}
\]

A slope-function \( c_L(t) \) is used:

\[
c_L(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{LT}, & 0 \leq t < LT \\ 1/2, & t \geq LT 
\end{cases}
\]

(3)

For sections III to V a phase pulse \( q(t) = c_1(t) \) is presumed (CPFSK-modulation).

![Fig. 1 Block diagram](image)

For simplification following notation is used for the phase function eq. 2 within the time interval \( mt \leq t < (m+1)t \), \( m \geq 0 \) (cf.[7]):

\[
\psi(t, \alpha) = 2\pi f_c t + \phi_0 + 2\pi h \sum_{i=0}^{m-1} a_i q(t - iT) + \sum_{i=m}^{m+L-1} a_i q(t - iT) + \sum_{i=0}^{m-1} a_i q(t - iT) + \sum_{i=m}^{m+L-1} a_i q(t - iT)
\]

(4)

with:

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reference frequency
\[ f_R = f_c - \frac{h(M-1)}{2} \]

phase-state
\[ q_{m} = (2\pi \eta / (\pi M - 1)) \mod 2\pi, \quad \eta = 0 \]

and phase transition function
\[ b(a, t) = \begin{cases} aq(t)-(M-1)c_L(t) & t < LT \\ 0 & t \geq LT \end{cases} \]

All frequency or phase deviations \( \Delta f \), \( \Delta \) are to be interpreted relative to the reference frequency \( f_R \) or reference phase

\[ q_{R}(t) = 2\pi f_R t + q_{0R} \]

For rational modulation indices \( h = p/q \) there are \( q \) different phase states \( \Theta \) relative to the reference phase \( \phi = \frac{M-1}{M} \) with no common divisor.

The optimization of the convolutional codes and the mapping rule is accomplished by aid of the minimum normalized squared Euclidean distance \((\mathrm{cf.}[2])\) between two signals \( s(t) \) for different source symbol series. The performance of the coding-modulation scheme disturbed by white Gaussian noise is described by this parameter with sufficient exactitude \((\mathrm{cf.}[8])\).

III. EXTRACTION OF THE MEMORY OF THE MODULATION PROCESS AND MAPPING RULES

For phase pulses \( q(t) = c(t) \) there solt M-q different phase trajectories within one time interval. Subsequently these phase trajectories will be called signal elements of the signal set. They are addressed by \( M \) different instantaneous frequency deviations \( \Delta f \) (phase transition factors \( a \)) and \( q \) phase states \( \Theta \) (phase at the beginning of the symbol interval). There are other methods for addressing the signal elements too: \( \Theta = (\Theta_1, \Theta_2, \cdots, \Theta_{M-1}) \).

Fig. 2 shows a graphic representation of the signal elements for the example \( M = 4, \quad h = 1/4. \)

The modulation process can be divided into a redundant encoder \( E_M \) with memory and rate \( r = \frac{1}{2}(M) \) that guarantees the phase continuity and a table of the signal elements. Within this encoder \( v = [14(q)] \) binary delay elements contain the influence of past signal elements on the phase state \( [x]; \) next integer \( \pm x \).

An example Fig. 3 shows such a desintegration of a CPFSSK-modulator for \( M = 4, \quad h = 1/4 \). The instantaneous frequency deviation is addressed by the binary symbols \( a, b \), the phase state by \( y_1, y_2 \). As the phase state is a function of all past symbols \( a \), \( \text{eq.} \), the encoder \( E \) has a recursive structure.

For CPFSSK all signal elements with the same frequency deviation - the elements of one row in Fig. 2 - are mapped to one codeword \( x \), which will be called "frequency-mapping." But there are many other possibilities. For a specific mapping rule the set of \( M \times q \) signal elements must be partitioned into \( M \) classes, each with \( q \) elements. The classes are mapped to the different modulator inputs \( X \). To guarantee phase continuity each class has to cover all possible phase-states. Thus, the elements of a class have to be selected all from different columns in Fig. 2. Therefore \( (M) \times (M-1) \) different possibilities exist for partitioning the signal elements into \( M \) classes. As \( M \) mapping rules exist between the codewords and the classes for one specific partitioning \((\text{cf.}[2])\), there are \( (M) \times (M-1) \) mapping rules in total.

If the restriction to frequency-mapping is omitted, the optimization problem for the encoder and the mapping rule gets enormous additional degrees of freedom. Although many of these mapping rules have identical properties \((\text{cf.}[2])\), it is quite impossible to take all cases into account.

IV. PHASE-STATE-MAPPING AND CONVOLUTIONAL CODES

In this section a mapping rule is considered for \( M = q \), in which the information is expressed by the phase at the end of a symbol interval, which is equivalent to the phase state \( \Theta \). The next symbol \( X \), i.e. \( (\Theta) \) for input symbol \( X \). The phase continuity is performed by the frequency deviation. Thus, phase and frequency change their roles as slave and master compared to frequency-mapping. (For the example of Fig. 2 this "phase-state-mapping" corresponds to a partitioning of the signal elements into classes, which is characterized by identical signs for the signal elements.)

For \( M = 1 \) the selection of a signal element is only a function of the actual and the preceding modulator input word \( X \). Thus, for phase-state-mapping the encoder \( E \) has a structure as indicated in Fig. 4. The part with memory is nonrecursive and linear in \( GF(2) \), whereas the memoryless (nonlinearly) combinatorial circuit depends on the special mapping rule and the method of addressing the signal elements.

If some output lines \( y \) to \( y_{(M-1)} \) of the encoder \( E \) address the instantaneous frequency deviation of the signal elements (the phase transition factors a respectively), a CPFSSK-мо-

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As for the image content, it appears to be a page from a technical or scientific document discussing the principles of phase modulation and frequency mapping in coding systems, with specific equations and diagrams illustrating the processes. The text is dense with technical terms and mathematical expressions, indicating a high level of detail and precision typical of such documents.
Fig. 3 Desintegration of a CFPFSK-modulator into an encoder $E_M$ with memory and a signal table ($N=4$, $h=1/4$)

$$
\begin{array}{c|c|c}
\text{signal table} & \text{signal table} \\
Y_0 & Y_1 & \theta \tilde{t} \\
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 2 \\
1 & 1 & 3 \\
\end{array}
$$

For phase-state-mapping the selection of a signal element is a function of the $n$ binary modulator input signals and their delayed versions. In the nonrecursive, obvious and minimal encoder realization the memory of the convolutional code (cf. [10]) is represented by $k$ shift registers, one for each encoder input, with total $v_k$ binary delay elements, where $k$ is the number of shift registers of the memoryless combinatorial circuit and the signal table (Fig. 4) can be generated from these $k$ shift registers using the generator polynomials of the convolutional encoder and variables multiplied by $D$. Thus, $v_k = k$ binary delay elements are sufficient for coding and modulation in such a canonical structure. Therefore $n-k$ binary delay elements are redundant for phase-state-mapping.

V. FREQUENCY-MAPPING AND CONVOLUTIONAL CODES

There is a disadvantage of phase-state-mapping compared to frequency-mapping, because for frequency-mapped signals only the differences between phase states have to be traced at the receiver, but no specific relation to the reference phase is necessary. So frequency-mapping rules have a natural property, which is comparable to the rotationally invariant of some coded PFSK systems [9]. From Fig. 2 it is recognizable that only for frequency-mapping rules the coordination of a signal element to a class is independent of the phase state. Perhaps there are special codes, with which rotationally invariant schemes arise for other mapping rules analogous to [9]. For the decomposition of the modulation process for frequency-mapping the structure of the encoder $E_M$ can be derived directly from eq. 6 ($t=1$). For this use the different phase-states $e_{\phi}$ are represented by integer numbers $0, 1, \ldots, (q-1)$. The next phase-state is the sum of the past phase state $e_{\phi} - 2$ and the non-negative integer

$$
p^t(q-1)+M-1)/2 \text{ with a subsequent reduction modulo } Q. \text{ Representing the phase-state by } v_k \text{ binary digits, the recursive structure given in Fig. 5 arises for the encoder } E_M.
$$

An obvious identification of redundant delay element within the CFPFSK-scheme combined with a convolutional codes is possible, if a part of this encoder $E_M$ is expressable by a switching circuit linear in GF(2), which is characterized by $g_q(D) = P(D)/Q(D)$ (In the example of Fig. 3 the output series $\psi_n$ is given by

$$
y_n(D) = x_n(D)P(D)/(1-D). \text{ If the generator polynomials of the minimal and obvious realization of the convolutional encoder } E_M \text{, which contributes to the inputs of the modulator with this linear phase-state memory, contain Q(D) (or factors of it) as factors, the corresponding phase-state address is generated by a noncanonical circuit and therefore redundant delay elements exist.}
$$

For the "natural" frequency-mapping rule (Q1 in [2]: increasing a mapped to increasing inputs $x$, interpreted as dual numbers), a linear description for the delay elements of the phase-state memory is possible for modulation indices $h = p/q$ with $q$ even ($p$ odd). For the LSB of $\theta_{m1}$ no carry handling and for $q$ even no reset operation due to the reduction modulo $q$ is necessary. The linear structure $B/(1+D)$ for the input $x_n$ of the modulator is shown in Fig. 6. Thus, all convolutional codes, for

43.1.3
which the polynomials generating the output \( x \) have even numbers of coefficients one, use one binary delay element in common with the modulation process.

VI. EXTENSION TO CPM WITH ARBITRARY PHASE PULSES

For \( L = 1 \), but \( q(t) \neq c_i(t) \) the only difference to the previous sections is that a phase transition function \( b(a,t) \) (eq.7) instead of an instantaneous frequency deviation is addressed by the phase transition factors \( \alpha \) or the address lines \( y_0 \) to \( y_{L-1} \) (cf. Fig. 3). By this means the representation of signal elements in Fig. 2 has a valid interpretation for arbitrary phase pulses too. The encoder \( E_M \) is not affected by the special shape of the phase pulses.

For partial response signals (\( L>1 \)) there are up to \( q^L \) different signal elements within one time interval \( T \). This signal elements can be addressed by the phase-state \( G \) and L phase transition factors \( b \) corresponding to the superposition of phase transition functions (eq.4).

For frequency-mapping the decomposition of the modulation process arises from eq.4 and is given in Fig. 7. The first nonrecursive part of the encoder \( E_M \) handles the superposition of phase pulses; the second recursive part belongs to the sequence of phase states and is independent of \( L \) (cf. eq.6). The outputs of the \( n(L-1) \) delay elements of the nonrecursive part can be generated from \( k(L-1) \) delay elements, added to the \( k \) shift registers of a preceding convolutional encoder with minimal and obvious realization (cf. section IV). Thus, \( (n-k)\cdot(L-1) \) delay elements of the combined encoder-modulator scheme are redundant within the nonrecursive part (cf.5). For \( q \) even, natural mapping, and generator polynomials with a even number of nonzero coefficients for the encoder output \( x \), one additional common delay element exists in the same way as for \( L=1 \).

For phase-state-mapping the encoder \( E_M \) consists of \( L \) encoders for \( L=1 \), each of them generating one phase transition factor \( \alpha \) of the address of the signal elements (Fig. 8). Thus \( (n-k)\cdot L \) delay elements of the combined encoder-modulator scheme are redundant. If a CPM-modulator is used instead of a signal table for the signal generation, the "mapper" \( E_M \) with memory (Fig. 4), which converts frequency-mapping to phase-state-mapping for \( L=1 \), performs this conversion for arbitrary phase pulses too (cf. Fig. 8).

VII. RESULTS

In table 1 to 3 some optimum convolutional codes and minimum squared Euclidean distances are listed for CPM with phase-state-mapping.
and frequency-mapping in octal numbers \((x_i, x_j)\) (cf. [2]). In the columns "phase-state-mapping" and "frequency-mapping with fusion" codes are given, which use \(L\) delay elements in common with the modulation process.

The columns "frequency-mapping without fusion" contain codes with \(L\) common delay elements. Therefore these codes, which are given for comparison and are partially taken from ref. [2], [3], [5], have one delay element less than the other codes. But the total number \(2^k\) of encodemodulator states is identical for each row of the tables. If no improvement is possible by a usage of the memory of the modulation process no code is listed.

For \(q(t)=c(t)\) (CPFSK), \(k=n/1=2, h=1/4, 4\), there are remarkable improvements of phase-state-mapping to frequency-mapping for \(v = 3, 5,\) and \(7\). (A free optimization of labeling the 16 signal elements to the pathes of a trellis with \(2^k\) states has given that for \(v = 3\) and \(v = 4\) there are better, but many equivalent schemes as long as phase continuity is presumed).

The usage of a common delay element for frequency-mapping gives gains for \(v = 4, v = 7\) and for \(h = 1/2\) for \(v = 2\). For the codes of rate 2/3 and CPFSK \(M = 8, h = 1/6\) the additional source symbol \(u_t\) is mapped without coding to \(x_i, (\text{cf. Fig. 4})\). In this case the total improvement of one additional delay element is possible by phase-state-mapping. Tables 2 and 3 contain results for partial response phase pulses of the type "raised-cosine" (1-RC cf. [1]). Approximately the same improvements are obtained by phase-state-mapping as for CPFSK (1REC). In one case almost the same minimum Euclidean distance is achieved with frequency-mapping. Therefore it is not clear, what is the prize for rotational invariance. (For 3 RC some reference values differs to those given in [5].)

TABLE 1: Some optimum codes and minimal Euclidean distances, frequency impulga: 1REC (q(t)=c(t))

<table>
<thead>
<tr>
<th>(v)</th>
<th>(M_6, h=1/2, r=1/2)</th>
<th>(M_6, h=1/4, r=1/2)</th>
<th>(M_8, h=1/8, r=2/3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(d_{min})</td>
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</tr>
</tbody>
</table>

\[\text{Table 1 from reference /2/3/}.

TABLE 2: Frequency Impulse q(t): 2RC

<table>
<thead>
<tr>
<th>(v)</th>
<th>(M_6, h=1/2, r=1/2)</th>
<th>(M_6, h=1/4, r=1/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(d_{min})</td>
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</table>

\[\text{Table 2 from reference /2/3/}.

TABLE 3: Frequency Impulse q(t): 3RC

<table>
<thead>
<tr>
<th>(v)</th>
<th>(M_6, h=1/2, r=1/2)</th>
<th>(M_6, h=1/4, r=1/2)</th>
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</thead>
<tbody>
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\[\text{Table 3 from reference /2/3/}.

REFERENCES


