Coordinated Digital Transmission: Theory and Examples
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Dedicated to Hans-Werner Wellhausen on the occasion of his 60th birthday

1. Introduction

Fast digital transmission over twisted pair lines, such as high-rate digital subscriber lines (HDSL) or asymmetric digital subscriber lines (ADSL), is a field of growing interest and large economic opportunities. Applying these techniques, new digital services (e.g., video on demand) can be provided cost-effectively to a wide clientele.

The main research efforts are to improve reliability and, at the same time, to increase the maximum field length for such transmission schemes. On the one hand possible strategies are the application of noise-whitening structures, trellis-coded modulation or signal shaping. On the other hand multiple-duplex transmission decreases bandwidth and hence avoids the use of frequency bands with high signal attenuation. In comparison with simplex transmission over single pair lines, dual-duplex systems increase the range of reliable communication considerably without the need of additional lines [1]. For HDSL systems dual-duplex transmission is state of the art.

Up to now, the two twisted pair lines are treated separately. Because of the geometry of the cables, especially if the star quad is used, one can expect that impairments are coupled into both lines in a similar way — the noise processes will be correlated. This statistical dependence of the noise processes can be exploited if the component schemes are “coordinated” in a certain way. Lechleider proposed a method of exploiting the correlation of simultaneous noise samples in dual-duplex transmission [2], [3]. But this strategy is not exhaustive, because correlation is spread in time by dispersion. Only the exploitation of the entire auto- and cross-correlation functions will give the maximum gain.

In this paper, we first introduce methods for exploiting the entire correlation. These schemes are comprised by the denomination “dynamic coordination”. It is shown that applying dynamic coordination only correlations of simultaneous noise samples or a spread in the signal-to-noise ratios (SNR) remain. Consequently, a so-called “static coordination” may achieve further improvements. Methods for static coordination are presented which are generalizations of [2], [3]. An important result shows that combining static and dynamic coordination channel capacity of the multi-channel can be approached as closely as for memoryless additive white Gaussian noise (AWGN) channels (a generalization of Price’s famous result [4]). Finally, realistic gains derived from crosstalk transfer function measurements are given exemplarily for a 2.048 Mbit/s HDSL.
transmission in the German Telekom network.

2. Dynamic Coordination

2.1 Discrete-Time Channel Model

Consider D-fold duplex transmission over twisted pair lines. At the receiver, it is possible to cancel crosstalk caused by the subsystems within the D transmission scheme, because the corresponding symbol sequences are known in the transceiver. Therefore, it is possible to use lines next to each other for fast digital communication. In the German Telekom network, especially the use of one star quad is considered for dual-duplex transmission. As in such schemes the main impairment is near-end crosstalk (NEXT), which is usually dominating over far-end crosstalk (FEXT), in this paper we assume noise solely caused by other inherent HDSL systems. This additive noise is assumed to be Gaussian (cf. [5], [6]) and (cyclo-)stationary. At the receiver, D parallel filters \( H_{R_k}(f) \), \( k \in \{1, \ldots, D\} \), for optimum linear zero-forcing are supposed, followed by \( T \)-spaced sampling [7]. This assumption does not restrict generality, because all other receiver concepts (e.g. matched filter, whitened filter) can be represented as a cascade of this continuous-time receiver input filter with a discrete-time filter [8].

The discrete-time model of the HDSL transmission taken into consideration consists of \( D \) parallel, independent schemes without inter symbol interference (ISI) or inter channel interference (ICI). Due to the \( T \)-spaced sampling, the additive, real noise sequences \( n_\mu(k) \), \( \mu \in \{1, \ldots, D\} \), where \( k \in \mathbb{Z} \) is the discrete time variable, are stationary and assumed to be Gaussian with zero mean. Hence, the auto- and cross-correlation sequences

\[
\varphi_{\mu\nu}(\kappa) = E \{ n_\mu(k + \kappa) n_\nu(k) \},
\]

\( \mu, \nu \in \{1, \ldots, D\} \)

are an exhaustive description. For conciseness, all signals throughout this paper are assumed to be real, because our examples deal with digital baseband transmission. The generalization of the results to equivalent complex signals for bandpass schemes is straightforward.

It is advisable to combine the \( \varphi_{\mu\nu}(\kappa) \) into a correlation matrix

\[
\Phi(\kappa) = [\varphi_{\mu\nu}(\kappa)] = E \{ n^T(k + \kappa) n(k) \},
\]

with

\[
n(k) = [n_1(k), \ldots, n_D(k)],
\]

and to take the z-transform of the normalized correlation matrix

\[
\Phi(z) = E \{ \Phi(\kappa) / \sigma_\Phi^2 \} = [ E \{ \varphi_{\mu\nu}(\kappa) / \sigma_\Phi^2 \} ] = \sum_{i=-\infty}^{\infty} \Phi_i z^{-i},
\]

whereby the variance \( \sigma_\Phi^2 \) is an arbitrary positive constant. Due to the properties of auto- and cross-correlation sequences (e.g. [9]) \( \Phi(z) \) is a hermitian matrix, i.e. \( \Phi^T(z) = z^{-1} \Phi(z) \).

Because \( \Phi(\kappa) \) (or equivalently \( \Phi(z) \)) is a sufficient description of the noise statistics, any model with \( D \) Gaussian processes and correlation matrix \( \Phi(\kappa) \) is equivalent. For our purpose, it is favourable to use \( D \) mutually uncorrelated, white, zero mean Gaussian processes with variance \( \sigma_\Phi^2 \), which are fed to a matrix filter \( F(z) \) which generates the desired noise statistics. It is easy to show that \( F(z) \) must be chosen to be

\[
\Phi(z) = F^T(z) F(z),
\]

where \( F(z) \) is the matrix filter, \( F(z) \) can be determined via spectral factorization (e.g. [5], [10]). Since the solution is not unique (given \( F(z) \) and an arbitrary constant unitary matrix \( X \), \( F(z) = X F(z) \) is another solution), we force the components of \( F(z) = \sum_{i=0}^{\infty} F_i z^{-i} \) to be minimum phase, stable and causal. Furthermore, \( F_0 \) is chosen to be a right upper triangular matrix.

If \( F(z) \) has a stable, causal inverse \( G(z) = F^{-1}(z) = \sum_{i=0}^{\infty} G_i z^{-i} \) it can be used as a whitening filter. From \( \left( \sum_{i=0}^{\infty} F_i z^{-i} \right) \left( \sum_{i=0}^{\infty} G_i z^{-i} \right) = I \) (identity matrix) \( G(z) \) can be determined via the recursive process

\[
G_0 = F_0^{-1}, \quad G_i = -G_0 \sum_{i=1}^{i} F_i G_{i-1}.
\]

The application of the whitening filter leads to the final discrete-time model with white noise processes but ISI and ICI, shown in Fig. 1. The vectors \( s(k) = [s_1(k), \ldots, s_D(k)] \) and \( e(r)(k) = [e_1(k), \ldots, e_D(k)] \) comprise the transmitted and received signals, respectively. \( n(k) \) is a vector with mutually uncorrelated, white Gaussian noise samples, each component with variance \( \sigma_\Phi^2 \).

The analog receiver filters \( H_{R_k}(f) \) and the noise whitening filter \( G(z) \) can be combined to a matrix
input filter. Since it is matched to the signals and produces white, mutually uncorrelated noise processes, it can be denoted as \textit{coordinated whitened matched filter} (CWMF) (cf. [11]) shown in Fig. 2 with

\[
H_{\text{CWMF}}(f) = \begin{bmatrix} H_{R1}(f) & 0 \\ \vdots & \ddots \\ 0 & H_{RD}(f) \end{bmatrix} G(e^{j2\pi f T}).
\]

(5)

2.2 Coordinated Decision-Feedback-Equalization

All known equalization techniques can be applied to the channel model shown in Fig. 1, especially coupled maximum-likelihood sequence estimation (MLSE) [11]. But unfortunately, a realization with the Viterbi-algorithm is impractical because of the huge number of states given by all possible combinations of the \(D\) sequences. With coordinated \textit{reduced-state sequence estimation} (RSSE) the expenditure can be lowered at the price of worse noise immunity. Other possible solutions are sequential decoding algorithms, such as M-algorithm or Fano-algorithm [8].

We concentrate ourself on sequence estimation with solely one state, i.e. only one survivor is determined per step. At each step a decision, e.g. with a simple slicer, is necessary, whereby the already decoded symbols are taken into consideration. This principle of sequence estimation is known as \textit{decision-feedback-equalization} (DFE), shown in Fig. 3 (cf. [12]).

For further optimization, constants \(V_\mu\) (comprised in a diagonal matrix \(V = \text{diag}(V_1, \ldots, V_D)\)) are incorporated in the transmitter in order to suitably distribute transmit power. The \(D\)-fold memoryless slicer generates an estimate for the transmit vector \(s'(k)\). Via the matrix filter \(B(z)\) ISI and ICI caused by \(G(z)\) are eliminated. The optimum filter \(B(z)\) can be determined in the noise free case and with the assumption that no decision errors were made, i.e. \(\hat{s}(k) = s'(k)\) (\(X(z)\) denotes the z-transform of \(x(k)\)):

\[
E''(z) = S''(z)VG(z) - \hat{S}(z)B(z) = S''(z) [VG(z) - B(z)].
\]

If zero-forcing DFE (ZF-DFE) is desired, \(e''_\mu(k)\) and \(s''_\mu(k)\) should only differ in a constant factor \(V_\mu G^0_{\mu\mu}\) \((G_0 = [G^0_{\mu\mu}])\). This condition leads to:

\[
VG(z) - B(z) \equiv V \text{diag} \left( G^0_{11}, \ldots, G^0_{DD} \right)
\]

or

\[
B(z) = V [G(z) - \text{diag} \left( G^0_{11}, \ldots, G^0_{DD} \right)] = \sum_{i=0}^{\infty} B_i z^{-i}.
\]

(6)

It is essential that only decisions already made are enlisted to compensate ISI and ICI. This demand corresponds to a causality in two different ways: First, the dependence only of previous decisions of all \(D\) channels, and second, a causality at time instant \(k\) relating to the ordering of the channels. Only the symbols \(\hat{s}_\mu(k)\), \(\mu \in \{1, \ldots, 1\}\) can be used to calculate \(e''_{\mu}(k)\). These requirements are fulfilled if \(F_0\) is a right upper triangular matrix with main diagonal elements equal zero. Because \(F_0\) is forced to be a right upper triangular matrix (and so this holds for \(G_0\), too), the demand is met (cf. [12]).

If there are no decision errors, \(D\) independent AWGN channels from \(s(k)\) to \(e''(k)\) with signal amplifications \(1\) and noise variances

\[
\sigma_\mu^2 = \sigma_s^2/G^0_{\mu\mu}^2
\]

result regardless the matrix \(V\) for unequal power distribution. In Section 3, a method for exploitation of the spread of the noise variances is presented.

2.3 Coordinated Noise Prediction

Coordinated DFE can be impractical like full MLSE if one of the components of \(G(z)\) has infinite impulse response. A common but not optimum approach is to
truncates the tails of these impulses. Here, we present an alternative way, related to topics in linear prediction theory, which leads to optimal filters for a given order. Transmission without and with noise whitening filter are special cases of the proposed scheme.

Fig. 4 shows the coordinated noise prediction [5], [13]. Noise measurements \( \hat{n}_\mu(k) \) can be created by subtracting the estimates \( \hat{s}_\mu(k) \) from the receiver input signal \( e_\mu(k) \):

\[
\hat{n}_\mu(k) = e_\mu(k) - \hat{s}_\mu(k).
\]

Since the error probability of the scheme has to be low for reliable transmission, nearly all estimates \( \hat{s}_\mu(k) \) will equal \( \hat{s}_\mu(k) \) and hence, the measurements \( \hat{n}_\mu(k) \) with \( n_\mu(k) \). Using the past measurements \( \hat{n}_\mu(k - \kappa) \), \( \kappa \in \mathbb{N} \), of all \( D \) channels noise estimates \( \hat{n}_\mu(k) \) can be derived via \( D \times D \) causal prediction filters \( P_{\mu\nu}(z) \) comprised in a matrix \( P(z) \). For FIR filters \( P_{\mu\nu}(z) = \sum_{\kappa=1}^{\infty} \rho_{\mu\nu}(\kappa) z^{-\kappa} \) of order \( p \) the predicted noise samples read:

\[
\hat{n}_\nu(k) = \sum_{\mu=1}^{D} \sum_{\kappa=1}^{p} \rho_{\mu\nu}(\kappa) \hat{n}_\mu(k - \kappa).
\]

Subtracting \( \hat{n}_\mu(k) \) from \( e_\mu(k) \), white, mutually uncorrelated residual noise processes \( n'_\mu(k) \) with minimum variances are desired. With the knowledge of the auto- and cross-correlation sequences \( \varphi_{\mu\nu}(\kappa) \) a minimum mean squared error (MMSE) criterion leads to the following, so called generalized Yule-Walker equations for determination of the tap weights \( \rho_{\mu\nu}(k) \) (see [5]):

\[
\sum_{\mu=1}^{D} \sum_{\kappa=1}^{p} \rho_{\mu\nu}(\kappa) \varphi_{\nu\nu}(\kappa - k) = \varphi_{\mu\nu}(k)
\]

with

\[
\mu, \nu \in \{1, \ldots, D\}, \quad k \in \{1, \ldots, p\}.
\]

The only difference compared to the usual scalar case [8], [14], [15] is that noise measurements of all \( D \) channels are used for prediction of noise in each channel. Because the cross-correlations are exploited, the mean squared error for a fixed order \( p \) will be smaller than in the case of \( D \) individual noise prediction structures.

It can be shown [5] that for a sufficient order \( p \) and small error probabilities only correlations of simultaneous noise samples remain:

\[
\varphi'(\kappa) = E\{n'_\mu(k + \kappa)n'_\nu(k)\} \approx 0, \quad \kappa \neq 0, \quad \mu, \nu \in \{1, \ldots, D\}.
\]

In Section 3, the further exploitation of these correlations is discussed.

### 2.4 Coordinated Tomlinson-Harashima Precoding

Coordinated DFE and noise prediction suffer from error propagation. But even worse, channel coding, e.g. trellis-coded modulation [16], cannot be applied in a straightforward manner, since DFE and noise prediction need zero delay decisions which is irreconcilable with the basic idea of channel coding.

To overcome these problems precoding at the transmitter can be used. In order to avoid an enhancement of transmit power, Tomlinson-Harashima precoding works with modulo arithmetics [17], [18]. This nonlinear operation is equivalent to a periodical extension of the signal constellation, i.e. a multiple representation of symbols by signal points. Symbol-by-symbol that signal point is chosen which minimizes the magnitude of the transmit signal. In a coordinated Tomlinson-Harashima precoder this is done for the entire transmit vector, see Fig. 5. A further generalization is trellis precoding which provides an additional shaping gain, i.e. a reduction of the average transmit power. For implementation, combined precoding/shaping techniques which need not scramble redundancy with data are of special interest. Representatives are shaping without scrambling or coordinated dynamics shaping [19] which beside a minimization of the dynamic range of the received signal offers various advantages for blind residual equalization at the receiver side.

### 3. Static Coordination

Applying dynamics coordination only correlations of simultaneous noise samples and/or a SNR spread of the individual channels remain. Further gains will be provided if these effects are exploited. As the methods developed in this section treat the signals only at the
same (discrete) time instant \( k \) we use the denomination “static coordination”.

### 3.1 Decorrelation and Capacity

Let \( \mathbf{n}(k) = [n_1(k), \ldots, n_D(k)] \) denote a vector of white, but mutually correlated Gaussian noise processes with zero mean. If dynamic coordination is applied, the \( n_\mu(k) \) will correspond to the residual noise processes \( n'_\mu(k) \). Under these assumptions the covariance matrix

\[
\mathbf{C} = \mathbb{E}\{\mathbf{n}(k)\mathbf{n}(k)\} = \phi(0) = \mathbb{E}\{n_\mu(k)n_\nu(k)\}
\]

(12)

comprises the entire noise statistics.

Our aim is to transform \( \mathbf{n}(k) \) in order to get \( D \) independent noise components \( n_\mu(k) \). This can be done by means of Karhunen-Loève expansion. For this purpose an eigenvalue decomposition of \( \mathbf{C} \) is necessary:

\[
\mathbf{C} = \mathbf{\Psi}^T \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_D
\end{bmatrix} \mathbf{\Psi},
\]

(13)

where \( \mathbf{\Psi} \) is the matrix of right eigenvectors of \( \mathbf{C} \).

Applying \( \mathbf{\Psi}^T = \mathbf{\Psi}^{-1} \) at the receiver \( D \) uncorrelated noise processes with variances given by the eigenvalues \( \lambda_\mu \) are obtained. In order to compensate this transform, a precorrelation of the transmit signals via the matrix \( \mathbf{\Psi} \) is necessary. \( D \) independent, non-dispersive equivalent channels with white, mutually uncorrelated noise processes and maximum SNR spread result. Fig. 6 illustrates the decorrelation. From top to bottom it is shown: continuous-time model of independent, parallel transmission; discrete-time model; applying pre- and decorrelation; resulting equivalent channels with mutually uncorrelated noise.

The advantage of this procedure can be explained by capacity arguments. Without decorrelation and coordination of the individual systems the performance is restricted by the largest noise variance \( \sigma^2 = \max_\mu \mathbb{E}\{|n_\mu(k)|^2\} \). Since the total signal power \( S_T \) will be distributed equally among the individual channels, the capacity in the uncoordinated case reads:

\[
C_u = \sum_{\mu=1}^{D} \frac{1}{2} \text{ld} \left( 1 + \frac{S_T/D}{\sigma^2} \right).
\]

(14)

For moderately high SNRs, the commonly used approximation

\[
C_u \approx \frac{D}{2} \text{ld} \left( \frac{S_T/D}{\sigma^2} \right)
\]

is used.

To maximize capacity of coordinated transmission a partitioning of the total transmit power according to the water-filling rule is optimum [5]:

\[
S_\mu + \lambda_\mu = \text{const.} \quad \text{with} \quad \sum_{\mu=1}^{D} S_\mu = S_T.
\]

(16)

At high SNRs \( S_\mu \approx S_T/D \) is true and the capacity for the coordinated transmission becomes:

\[
C_c \approx \sum_{\mu=1}^{D} \frac{1}{2} \text{ld} \left( 1 + \frac{S_T/D}{\lambda_\mu} \right) \approx \frac{D}{2} \text{ld} \left( \frac{S_T/D}{\sqrt[\mu]{\prod_{\mu=1}^{D} \lambda_\mu}} \right).
\]

(17)

A comparison of eqs. (15) and (17) shows the gain of static coordination:

\[
G_c \approx \max_\mu \mathbb{E}\{|n_\mu(k)|^2\}.
\]

(18)

Since \( \sqrt[\mu]{\prod_{\mu=1}^{D} \lambda_\mu} \leq \sqrt{|\mathbf{C}|} \leq \max_\mu \mathbb{E}\{|n_\mu(k)|^2\} \) (cf. [20]), \( G_c \geq 1 \) is valid, i.e. coordination will always provide a gain in capacity\(^1\).

The methods and results given above are extensions of those in [2], [3] which are restricted to \( D = 2 \).

For implementation static coordination can be incorporated into dynamic coordination. At the noise prediction structure this can be easily done by the choice \( \rho_{\mu\nu}(0) \neq 0 \) for \( \mu \neq \nu \), and applying the generalized Yule-Walker equations where the limits of the sums have to be modified accordingly [5]. This direct method produces independent channels with mutually uncorrelated noise samples and maximum SNR spread. For coordinated DFE with optimum filters this statement holds anyway.

\(^1\)\(|\mathbf{C}|\) denotes the determinant of matrix \( \mathbf{C} \).
3.2 Optimization of the Signal Constellation

The channel capacity is the ultimate bound for reliable transmission. But unfortunately, it doesn’t tell us how to choose the transmission scheme. For this purpose we optimize a $D$-dimensional signal constellation taken from the scaled lattice $\mathbb{Z}^D$ for maximum minimum Euclidean distance. The uncoordinated reference system with total rate $R_T$ is assumed to have the same signal constellation \(\{\pm 1, \pm 3, \ldots, \pm (2^{R_T/D} - 1)\}\) in each individual channel (dimension). For coordinated transmission $V_\mu = \{\pm 1, \pm 3, \ldots, \pm (2^{R_T} - 1)\}$ is assumed to be the constellation in channel $\mu$ with factor $V_\mu$ for optimization and $R_\mu$ the individual rate. The constraints for optimization are on the one hand a fixed total rate

\[
R_T = \sum_{\mu=1}^{D} R_\mu \quad (19)
\]

and on the other hand a fixed total transmit power $S_T$ for the $D$ channels. Thereby a dense lattice is assumed and the method of continuous approximation [21] is applied. Under these assumptions the total transmit power is given by

\[
S_T = \sum_{\mu=1}^{D} S_\mu = \sum_{\mu=1}^{D} \frac{1}{3} V_\mu^2 2^{2R_\mu} = \frac{1}{3} D 2^{2R_T/D} \quad (20)
\]

The last equation considers that $S_T$ should be equal to that of uncoordinated transmission.

It is reasonable that for an optimum design, all individual channels perform with same error rate. As the quadratic Euclidean distance relative to the noise variance is the essential parameter for noise immunity

\[
d_\mu^2 = \frac{4V_\mu^2}{\lambda_\mu} = d^2 = \text{const.} \quad \forall \mu \quad (21)
\]

should be designed.

Inserting eq. (21) into eq. (20) and solving for $d^2$ leads to

\[
d^2 = \frac{4D 2^{2R_T/D}}{\sum_{\mu=1}^{D} \lambda_\mu 2^{2R_\mu}} \quad (22)
\]

Since this distance has to be maximum, the denominator in eq. (22) must be minimized under the constraint eq. (19). Lagrange optimization [22] results:

\[
\lambda_\mu 2^{2R_\mu} = \text{const.} \quad \forall \mu \quad (23)
\]

Taking the product over all terms in eq. (23) yields:

\[
\prod_{i=1}^{D} \lambda_i 2^{2R_i} = \prod_{i=1}^{D} \lambda_i \prod_{i=1}^{D} 2^{2R_i} = 2^{2R_T} \prod_{i=1}^{D} \lambda_i = (\lambda_\mu 2^{2R_\mu})^D \quad (24)
\]

Regarding the last terms in eq. (24) and solving for $R_\mu$, the best partitioning of $R_T$ into rates $R_\mu$ for the individual channels reads:

\[
R_\mu = \frac{R_T}{D} + \frac{1}{2} \log_2 \left( \prod_{i=1}^{D} \frac{\lambda_i}{\lambda_\mu} \right) \quad (25)
\]

If $R_\mu < 0$ for some $\mu$, eq. (25) has to be calculated once again excluding those channels. Eq. (25) leads to a continuous distribution of $R_T$ among the channels. This can be closely approximated if channel coding with proper rates is applied.

The signal power of the individual channels can be determined applying eqs. (21) and (23):

\[
S_\mu = \frac{1}{3} V_\mu^2 2^{2R_\mu} = \frac{1}{12} \lambda_\mu 2^{2R_\mu} = \text{const}. \quad (26)
\]

Thus, the total transmit power should be distributed equally among those channels with $R_\mu > 0$. The number of these channels is denoted by $D'$. The optimum partitioning of the total transmit power into powers $S_\mu$ for the individual channels reads:

\[
S_\mu = S_T/D'. \quad (27)
\]

Substituting eq. (25) into eq. (22) with some elementary calculations the optimum quadratic Euclidean distance relative to the noise variance reads:

\[
d^2 = \frac{4}{\sqrt{\prod_{\mu=1}^{D'} \lambda_\mu}} \quad (28)
\]

Compared with uncoordinated transmission (no decorrelation and $R_\mu = R_T/D$), $S_\mu = S_T/D$, where the worst case for $d^2$ is $4/\max_\mu E\{|n_\mu(k)|^2\}$, the coordination gain is given by

\[
G_c = \max_\mu E\{|n_\mu(k)|^2\} \quad (29)
\]

Notice that eq. (29) is identical to the result derived with capacity arguments (eq. (18)). This means that the gain in capacity can exactly be exploited by optimum power and rate allocation with constellations chosen from the lattice $\mathbb{Z}^D$. Application of coordination will always provide a gain – in no situation a loss occurs.

Further improvements are possible if no square constellations are used, but lattices which are optimum in $D$ dimensions. For $D = 2$, this would be a hexagonal constellation with a sphere boundary. The gain of these constellations relative to those chosen from the lattice $\mathbb{Z}^D$ with a cubic boundary is – like all coding and shaping gains – simply additive (in dB), see [23], to the coordination gain given by eq. (29). Simple methods for achieving hexagonal constellations via rotated signal constellations are given in [5], [24].

4. Coordinated Transmission and Capacity

An important result [4] in digital transmission theory is that the capacity of ISI channels can be achieved
asymptotically by shaping and coding developed for the AWGN channel if ideal (i.e. errorfree) DFE is applied. We will show that this statement holds also for the matrix channel with ISI and ICI if the individual systems are coordinated in the optimum way given in the last sections.

After applying dynamic coordination only simultaneous noise samples are correlated. Via pre- and decorrelation channels with independent noise processes and maximum SNR spread are provided. In the last step, this spread is averaged by optimum power and rate distribution — $D$ identical AWGN channels result (regardless of the rate which they are supporting).

Consequently, for coordinated DFE with optimum feedback and feedback filters the SNR incorporating static coordination reads

$$SNR = \frac{S_T/D}{\sigma_0^2 \left( \prod_{\mu=1}^{D} G_{\mu0}^0 \right)^{2/D}} \cdot \prod_{\mu=1}^{D} G_{\mu0}^0$$

(30)

if $\lambda_\mu = \sigma_0^2 \left( G_{\mu0}^0 \right)^{-2}$, see eq. (7), and $\Psi = I$ (I: identity matrix) is considered:

Since $[G_{\mu0}^0] = G_0 = F_0^{-1}$ and $F_0$ are right upper triangular matrices, $\prod_{\mu=1}^{D} G_{\mu0}^0 = |F_0|^{-1}$ is valid and the SNR results in:

$$SNR = \frac{S_T/D}{\sigma_0^2 |F_0|^{-2/D}}.$$  

(31)

In order to calculate $|F_0|$ we have to factorize

$$|\Phi(z)| = |F^T(z)F^*(z^{-1})| = |F(z)||F^*(z^{-1})|.$$  

For $z = e^{i2\pi fT}$, $\ln|\Phi(e^{i2\pi fT})|$ is a periodic function in $f$ and hence can be expressed as a Fourier series:

$$\ln|\Phi(e^{i2\pi fT})| = \ln|F(e^{i2\pi fT})| + \ln|F^*(e^{i2\pi fT})| = \sum_{i=-\infty}^{+\infty} c_i e^{-i2\pi i f T}$$

(32)

with

$$c_i = T \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \ln|\Phi(e^{i2\pi fT})| e^{-i2\pi i f T} df.$$  

(33)

At the same time the terms for $i \geq 0$ in eq. (32) are the Taylor series expansion of $\ln|F(z)|$, because the components of $F(z)$ are forced to be causal. Replacing $e^{i2\pi fT}$ again by $z$ and splitting $c_0$ between the causal and anticausal part, it leads to:

$$\ln|F_0 + F_1 z^{-1} + \cdots| = c_0/2 + c_1 z^{-1} + \cdots$$

(34)

For $z^{-1} \to 0$, $\ln|F_0| = c_0/2$ results and eq. (31) can be expressed by

$$SNR = \frac{S_T/D}{\sigma_0^2} \exp \left\{ -\frac{T}{D} \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \ln|\Phi(e^{i2\pi fT})| df \right\}.$$  

(35)

The symbol error rate for uncoded $M$-ary transmission can be calculated by

$$P_e = \frac{2(M-1)}{M} Q \left( \frac{\sqrt{3SNR}}{\sqrt{M^2-1}} \right)$$

(36)

where $Q(z) = (1/\sqrt{2\pi}) \int_{-\infty}^{+\infty} \exp(-t^2/2) dt$ is the complementary Gaussian integral function and $M = 2^{R_T/D}$. Following the argumentation of [4], the SNR required for $P_e = 10^{-5} \approx Q(\sqrt{18})$ is approximately $6 \cdot 2^{2R_T/D}$. Hence, the transmission rate per seconds is given by

$$R = \frac{R_T}{T} \approx \frac{D}{2T} \log \left( \frac{SNR}{6} \right).$$  

(37)

Inserting the results above, for high SNR the rate [bit/seconds] reads:

$$R \approx \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \left[ \left( \frac{S_T/D}{\sigma_0^2} \right)^D \Phi^{-1}(e^{i2\pi fT}) \right] df.$$  

(38)

This result has to be compared with the capacity of the multi channel. Let $S(f)$ denote the matrix of transmit power spectra and $N(f)$ the matrix of auto- and cross-power spectral densities of the additive Gaussian noise, the capacity is given by [20]:

$$C = \int_{-\infty}^{+\infty} \log|I + S(f)N^{-1}(f)| df.$$  

(39)

In [4], it is shown that in the case of strictly monotonous decreasing channel attenuation (which is true for the HDSL environment), a constant power density in the first Nyquist set of frequencies $f \in \left[-\frac{1}{2T}, \frac{1}{2T}\right]$ is optimum. Regarding eq. (16) for multi channels $S(f) = T^2 \sigma_0^2 I$, $|f| \leq \frac{1}{2T}$ has to be chosen for high SNRs. Here, $S_c$ denotes the total transmit power for achieving capacity. Under this assumption and because of the design of the receiver input filters (see Section 2) $N(f) = T \sigma_0^2 \Phi(e^{i2\pi fT})$, $|f| \leq \frac{1}{2T}$, and zeros otherwise. Thus, for high SNRs the capacity reads:

$$C \approx \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \left[ \frac{S_c/D}{\sigma_0^2} \right]^D \Phi^{-1}(e^{i2\pi fT}) df.$$  

(40)

For $R = C$, $S_c = 6ST$ results, which is the same statement as derived in [4] for a single ISI channel. Therefore, transmission over parallel channels with mutually correlated noise processes can achieve capacity with powerful coding and shaping techniques if coordinated DFE with optimum rate and power distribution is applied. Starting from DFE, the gap between
error performance of the uncoded transmission and capacity of the multi channel cannot solely be bridged by coding and shaping on their own, but only together with the methods of coordinating the individual channels in one of the ways derived in this paper. This is a generalization of Price’s famous result [4].

5. Examples for HDSL Transmission

5.1 Calculation of Correlations

In order to show the gains one can expect in practice by applying coordinated transmission, the noise correlation sequences have to be known. They can be determined e.g. by crosstalk measurements, but this requires a great deal of work and is only valid for a specific situation. Consequently, we present a more flexible way of determining correlation sequences via cable and crosstalk transfer functions. They can easily be measured for a multi-pair cable and every system environment can be studied starting from this knowledge.

As explained above, only NEXT caused by identical systems is assumed. Therefore, the impairment in channel $\mu$ is the sum of the transmitted sequences $\langle a_q(k) \rangle$, $q \in \{1, \ldots, Q\}$ of all Q disturbing systems in the cable bundle, each filtered with the individual transmit spectrum $G_T^q(f)$, the crosstalk transfer function $H_{Xq\mu}(f)$ and the receiver input filter $H_R^\mu$. The fact that the sequences $\langle a_q(k) \rangle$ may not be synchronous can be incorporated in the NEXT impulse response via delays $\vartheta_q \in [0, T)$. Fig. 7 shows the crosstalk model for the channels $\mu$ and $\nu$ with $(F^{-1}$ denotes the inverse Fourier transform)

$$h_{q\mu}(t) = F^{-1}\{H_{q\mu}(f)\}$$

and

$$H_{q\mu}(f) = G_T^q(f)H_{Xq\mu}(f)H_R^\mu(f).$$

With the denominations explained above the noise samples can be written as:

$$n_{q\mu}(k) = \sum_{q=1}^{Q} \sum_{k=-\infty}^{+\infty} a_q(k)h_{q\mu}((k - \kappa)T - \vartheta_q). \quad (43)$$

Since after $T$-spaced sampling stationary noise sequences result, auto- and cross-correlation sequences can be derived by some calculations:

$$\varphi_{q\mu}(\kappa) = E\{n_{q\mu}(k + \kappa)n_{q\nu}(k)\} = \sum_{q=1}^{Q} \sum_{q=1}^{Q} \sum_{k_1}^{k_2} E\{a_{q_1}(k_1)a_{q_2}(k_2)\} \cdot h_{q_1\mu}((k_1 + \kappa)T - \vartheta_{q_1})h_{q_2\nu}(kT - \vartheta_{q_2}). \quad (44)$$

As we assume the data sequences $\{a_q(k)\}$ to be i.i.d. and independent of each other, the expected value in eq. (44) becomes $E\{a_{q_1}(k_1)a_{q_2}(k_2)\} = \sigma_{q_1}^2$ for $k_1 = k_2$ and $q_1 = q_2$ and zero otherwise. Hence, the correlation sequences read:

$$\varphi_{q\mu}(\kappa) = \sum_{q=1}^{Q} \sigma_{q\mu}^2 \sum_{k} h_{q\mu}((k + \kappa)T - \vartheta_q)h_{q\nu}(kT - \vartheta_q). \quad (45)$$

5.1.1 Plesiochronous Interference

In the case of plesiochronous interference the phases $\vartheta_q$ are unknown and hence assumed to be equally distributed in $[0, T)$. The correlation sequences $\varphi_{q\mu}(\kappa)$ can be determined by time averaging of $\varphi_{q\mu}(\kappa)$:

$$\varphi_{q\mu}(\kappa) = \frac{1}{T}\int_{0}^{T} \cdots \int_{0}^{T} \varphi_{q\mu}(\kappa) \, d\vartheta_1 \cdots d\vartheta_Q. \quad (46)$$

Some elementary calculations lead to:

$$\varphi_{q\mu}(\kappa) = \frac{1}{T} \sum_{q=1}^{Q} \sigma_{q\mu}^2 \int_{-\infty}^{+\infty} h_{q\mu}(\kappa T - \tau)h_{q\nu}(\tau) \, d\tau = \frac{1}{T} \sum_{q=1}^{Q} \sigma_{q\mu}^2 \left[h_{q\mu}(\kappa T) \ast h_{q\nu}(-\kappa T)\right]_{\tau=-\kappa T}. \quad (47)$$

For simplification we assume identical NEXT producing systems, i.e. $G_T^q(f) = G_T(f)$ and $\sigma_{q\mu}^2 = \sigma_\mu^2$, $\forall q$. Fourier transformation of eq. (47) and considering eq. (41) leads to the auto- and cross-power spectral densities of the crosstalk noise:

$$\Phi_{q\mu}(e^{j2\pi f/T}) = \frac{\sigma_\mu^2}{T^2} \sum_{q=1}^{Q} \sum_{l=-\infty}^{+\infty} H_{q\mu}(f - l/T)H_{q\nu}^*(f - l/T). \quad (48)$$

Notice that the sampling process in eq. (47) leads to spectral periodic continuation.

\footnote{To simplify notation $\sum_{k}$ is written instead of $\sum_{k=-\infty}^{+\infty}$.}
5.1.2 Synchronous Interference

If the NEXT causing systems run synchronous which is true e.g. at the central office, there is the sampling phase at the receiver as a free parameter which is the same for all systems, i.e. $\vartheta_q = \vartheta$, $\forall q$. Inserting into eq. (43) gives:

$$\phi_{\mu \nu}(r) = \sum_{q=1}^{Q} \sigma_q^2 \left( h_{\mu \nu}(r) * h_{\nu \mu}(-r) \right)$$  \hspace{1cm} (49)

with $h_{\nu \mu}(r) = h_{\mu \nu}(kT - \vartheta)$.

In contrast to eq. (47) here $h_{\mu \nu}(t)$ is first sampled and then convolved instead of sampling the result of the convolution.

For identical systems, as described above, the power densities can be written regarding the properties of the Fourier transform and eqs. (41), (42):

$$\Phi_{\mu \nu}(e^{i2\pi f/T}) = \frac{\sigma_q^2}{T^2} \sum_{q=1}^{Q} \sum_{l_1} H_{\mu \nu}(f - l_1/T) e^{i2\pi l_1 \vartheta / T} \cdot$$

$$
\cdot \sum_{l_2} H_{\nu \mu}^*(f - l_2/T) e^{-i2\pi l_2 \vartheta / T}. \hspace{1cm} (50)
$$

5.2 Results

The results shown in this section are valid for a HDSL system with 2.048 Mbit/s which is standard in Europe. Uncoded dual-duplex transmission and the linecode 2B1Q (quaternary baseband transmission) are assumed. Hence, the symbol frequency is $1/T = 512$ kHz. Since we assume exclusively NEXT caused by other, identical HDSL systems (self-NEXT) the transmit pulse $G_{\mu \nu}(f) = G_T(f)$ and the transmit power has not to be specified (cf. [1]).

The cable attenuation and the NEXT transfer functions were measured with a network analyzer for twisted pair lines with 4 km field length. The line diameter is 0.4 mm (close to 26AWG) which is the only one used for subscriber lines in the range up to 4 km in the German Telekom network [25]. For worst case statements, NEXT only within a binder group consisting of 5 star quads (numbered by #1/2, #3/4, #5/6, #7/8, and #9/0, respectively) is considered. For the presented numerical results the star quad #1/2 is used for data transmission and the star quads #3/4 and #7/8 for the NEXT producing signals.

5.2.1 SNRs and Correlations

At first, the SNRs $\sigma_q^2/\phi_{\mu \nu}(0)$ after optimum linear zero-forcing equalization for the two lines are given, assuming that the data signals are sampled at the optimum instants. Fig. 8 shows the SNRs in the pairs #1 and #2 for synchronous interference versus the normalized sampling phase $\vartheta$. The two additional straight lines are valid for the plesiochronous situation, i.e. averaging over all possible relative phases of the four disturbing signals (cf. eq. (46)). The SNRs are relatively high, because the measured line attenuation is moderate (especially at high frequencies) compared to the usually used worst case approximations of [26]. But for an uncoordinated transmission, the pair with the lower SNR still limits system performance.

In order to calculate the gains by coordination, we need the auto- and cross-correlation sequences. For plesiochronous and synchronous interference ($\vartheta/T = 0.2$, 0.45 and 0.7, respectively) they are shown in Fig. 9. A quaternary constellation $\{\pm 1, \pm 3\}$ is expected and hence $\sigma_q^2 = 5$. It is noticeable that for phases $\vartheta$ with high SNR the correlations are low and vice versa. As the gain grows with increasing correlation, one can expect that the strong fluctuation over $\vartheta$ (cf. Fig. 8) of the resulting SNRs are smoothed. Please notice that for an application of static coordination (cf. [2], [3]) only the correlations and the SNR spread marked with solid dots are utilized. To exploit the entire course, a dynamic coordination procedure has to be used.

5.2.2 Prediction and Coordination Gains

As already mentioned, the SNR is improved by noise prediction structures. This effect is shown in Fig. 10 for plesiochronous interference versus the order $p$ of the prediction filters. In each case the curves show the minimum SNR of both channels (pairs #1 and #2). The dash-dotted line is valid for individual, independent noise prediction structures for each channel without static coordination. The solid line corresponds to coordinated noise prediction (Section 2.3) in combination with optimum static coordination (gain given by eq. (29)). Furthermore, the maximum achievable SNRs ($p \to \infty$) are plotted (dashed lines). By the methods presented in this paper, additional gains of about 2 dB over independent reception – the state of the art for HDSL transmission – are possible for the given example.

Finally, the asymptotically achievable SNRs for independent reception and for dynamic coordination are plotted in Fig. 11 versus the sampling phase. For ref-
ference, the minimum SNRs after linear zero-forcing equalization are displayed, too. With state-of-the-art methods, SNRs between 38.26 and 42.16 dB are possible. Dynamic coordination provides further improvements of 1.54 to 2.82 dB. For plesiochronous interference the SNR of 30.23 dB is increased to 2.34 dB. These gains correspond to an additional field length of about 180 to 340 m [1].

The results of various crosstalk situations are summarized in Table 1. The number of crosstalk producing signals ranges from 8 (the whole binder group is used for HDSL transmission) down to 2. In each case, the SNR for independent reception with individual noise prediction \( p \rightarrow \infty \) and the gain due to coordination are shown. For synchronous interference the minimum and maximum values versus the sampling phase are given.

Fig. 9. Auto- and cross-correlation sequences for plesiochronous and synchronous interference.

Fig. 10. Signal-to-noise ratios applying noise prediction structures versus the prediction order \( p \).

Fig. 11. Asymptotic signal-to-noise ratios applying noise prediction versus sampling phase.

It is evident, that the less the number of interfering signals the higher are the SNRs. But please notice, the additional gains by coordination increase, too. Thus, lines best for HDSL transmission may be further improved by coordination, whereas it is not rewarding in the case of high interference. This result is reasonable because the more crosstalk causing systems are active, the more the cross-correlations are averaged and hence lowered. The most gain is possible if only one dominating disturber is present. Consequently, coordinated digital transmission is a method not suitable for the use in cables which are covered exclusively with HDSL transmission, but in such situations where only some, very reliable systems over long distances are desired. The advantage of coordination is that it is always superior to independent reception – never a loss occurs. The results and statements derived from crosstalk transfer functions were confirmed by a great amount of NEXT measurements performed at the Research Center of the German Telekom, Darmstadt.

6. Conclusions

Methods for the exploitation of the entire noise correlation sequences in multiple signal receivers were proposed. A possible, low expense structure is generalized noise prediction with optimum filters for a given order. Choosing optimum power and rate distributions the capacity of the multi channel can asymptotically be achieved by channel coding methods. Examples for HDSL transmission showed the possible gains relative to the state-of-the-art concepts. Further, a method for determining the correlation sequences from the crosstalk transfer functions was presented.

All in all, the additional system complexity applying coordinated digital transmission is moderate and well invested compared to the gains which are possible. Especially for isolated high rate data transmission schemes the proposed methods can improve reliability and/or increase the maximum field length.
Table 1. SNRs for independent reception with individual noise prediction of infinite order and gains due to coordination – synchronous and plesiochronous interference.

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<th>1/2</th>
<th>3/4</th>
<th>5/6</th>
<th>7/8</th>
<th>9/0</th>
<th>Synchronous SNR min/max (dB)</th>
<th>G min/max (dB)</th>
<th>Plesiochronous SNR (dB)</th>
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