An Analytic Solution to the Performance of Iterated Soft Decision Interference Cancellation for Coded CDMA Transmission over Frequency Selective Fading Channels

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Abstract — In this paper coded transmission over time-variant multipath Rayleigh-fading channels employing Direct Sequence Code Division Multiple Access (DS–CDMA) is considered. Assuming ideal knowledge of the actual channel state and randomly chosen spreading sequences, the performance of iterative multiuser equalization based on adapted MMSE-filters combined with serial successive cancellation and single user decoding is solved analytically. For this, merely the statistics of the decoder must be known. Equipped with this solution, cumbersome simulations of the whole multiuser system can be avoided to evaluate the achievable bit error ratio in a wide range of interesting parameters.

I. INTRODUCTION

The search for feasible multiuser receivers applicable to transmission with DS–CDMA has attracted considerable interest due to the deployment of the 3rd generation mobile telecommunications system and first studies with respect to the 4th generation.

While for low system loads \( \beta = K/N \ll 1 \) (K denotes the number of users and N the spreading factor) the use of linear receivers based on matched filter banks is sufficient, for high system loads \( \beta \approx 1 \) more sophisticated receiver algorithms are required instead. Among the last one it turned out that iterative schemes using decision-feedback allow to reach near single user performance while possessing manageable complexity [1, 2, 3, 4, 5, 6]. In all these studies the performance of the iterative scheme is evaluated by means of simulations.

The aim of this paper is to provide an analytic solution to the achievable Bit Error Ratio (BER) for coded multiuser transmission over frequency selective fading channels to a common receiver. In order to achieve near single user performance with a relatively low complexity and delay, especially small number of iterations, we restrict ourselves to the application of MMSE-filters, single user decoding with maximum a posteriori symbol by symbol estimation (MAP–SSE) and serial cancellation at the receiver.

So, a recent result on the limiting signal to noise ratio at the output of an MMSE-multiuser interference suppression filter for random spreading sequences in the case \( N \gg 1 \) can be used [7]. Our investigation was also motivated by the analysis of Turbo–Codes for transmission over additive white Gaussian noise (AWGN) channels given in [8] incorporating numerical evaluation of the extrinsic information’s variance provided by the MAP–SSE component decoders.

As to cope with the multipath fading channel implying unequal, time-variant received powers of the users, an equivalent transmission model has to be employed. Trying to emulate the actual receiver’s decision chain by means of a mathematical model, it turns out that the noise variance at the output of the interference suppression filter is one key parameter. To find a suitable formula for this variance is one of the main problems dealt with in this work. Equipped with the noise power and our equivalent channel model, the resulting BER can be obtained immediately. An additional burden is imposed by the serial cancellation carried out. As a consequence, the expected average multiuser interference differs for all users even within one iteration cycle. To devise a reasonable method for approximation of the actual soft decisions of each user is another point being addressed in this paper.

The prediction of the performance of a multiuser system for frequency selective fading channels is one goal. The other purpose of this paper is to verify that the advantages of nonorthogonal transmission compared to orthogonal multiple access schemes indicated by an information theoretical approach to this problem [9] can be realized in practical systems, too.

The paper is arranged as follows. In Section II, the transmission model is given and motivated. The receiver algorithm under study is described in Section III. The analytic solution for the BER reached by the receiver is derived in Section IV. A comparison of the actual performance of the receiver and the analytic results is given in Section V. Finally, Section VI points out conclusions.

II. TRANSMISSION MODEL

Coded transmission of \( K \) users over frequency selective fading channels with DS–CDMA to a single receiver is considered. The underlying discrete-time equivalent complex baseband transmission model is illustrated in Fig. 1a).

![Figure 1: a) Transmission model of CDMA system with \( K \) users b) Tapped-delay-line channel model with \( L \) paths.](image-url)

Here, \( x_k \{ \mu \} \) and \( s_k \{ \mu \} = (s_{1,k} \{ \mu \}, \ldots, s_{N,S,k} \{ \mu \})^T \) denote the
kth user's transmitted channel symbol and his/her spreading sequence in the mth transmission interval, respectively. The users’ channel symbols are chosen from the set \( x_k \in X_k \). They are obtained by convolutional encoding of the kth user’s data sequence \( d_k \) and subsequent passing of the codeword through a random interleaver \( x_k \). The elements of the unit energy spreading sequence are drawn randomly as \( s_{k,i}[\mu] \in \{ \pm 1 \} / \sqrt{N_s} \) \( \forall \mu, j \). The variables \( \tau_1, \ldots, \tau_k, \tau_{k+1} \in [0, \ldots, N-1] \) represent the users’ delays. The kth user’s modulated sequence \( x_k[n]\sigma_n, [\mu], \ldots, x_k[n]\sigma_{N-1}, [\mu] \) is transmitted over a linear dispersive channel with discrete time-variant weight function \( h_k[n] \). In order to account for the multi-path propagation the well known tapped-delay-line channel model (see Fig. 1b) assuming L resolvable paths is used, \( T_c \) is the length of one chip interval. As usual, it is assumed that the path weights \( h_k[j], n = 1, \ldots, L, k \) are independent and zero mean proper complex Gaussian distributed with variance \( \sigma_n^2 = \mathbb{E} \left[ \left| h_n[n] \right|^2 \right] \), \( \forall k \). For clarity and comparability of the results presented, throughout this paper the study is restricted to the so-called equal (average) gain channel, i.e., \( \sigma_n^2 = 1/L \), \( \forall k \). Supposing that the path gains are constant over one channel symbol the received signal resulting from \( x_k[n] \) can be written as \( x_k[n] s_k[n], x_k[n], \ldots, x_k[n] s_{N-1}, n \). Where \( s_k[n] = (x_k[n] s_k[n], x_k[n], \ldots, x_k[n] s_{N-1}, n) \) is the kth user’s effective spreading sequence. It is obtained by convolution of the actual spreading sequence \( s_k[n] \) and \( h_k[n] \), i.e., \( s_k[n] = \sum_{n=-\infty}^{n=N} h_k[n] s_k[n], j = 1, 2, \ldots, N = N + L - 1 \). In the following the focus is on the behavior of this multiple access system when \( N \) is large compared to the number of resolvable propagation paths \( L \), i.e., \( N \gg L \). In addition, for sake of analytical tractability synchronous transmission is supposed, i.e., \( \tau_0 = 0, \forall k \). Hence, inter-symbol interference is of minor importance in this situation and the received signal can be modeled as

\[
y[n] = S'[n][x[n]] + n[n]. \tag{1}
\]

The vectors \( y[n] = (y_1[n], \ldots, y_{N}', [n]) \) as well as \( n[n] = (n_1[n], \ldots, n_{N}', [n]) \) represent the received signal and the additive channel noise, respectively. The i.i.d. samples \( n_j[n], 1 \leq j \leq N', \) are zero mean complex Gaussian random variables with variance \( \sigma_n^2 \). Further, \( x[n] = (x_1[n], \ldots, x_{N}', [n]) \) and \( S'[n] = (s_1[n], \ldots, s_{N}', [n]) \) consist of the K users’ transmitted symbols and their effective spreading sequences as well.

Before proceeding with the description of the receiver and its analytic treatment, it has been emphasized that the restriction to the model given in Eq. (1) is merely made in order to gain theoretical insight. So, the random choice of spreading sequences leads to virtually the same performance for the general asynchronous case, too (cf. [10]). This is confirmed by our own studies.

III. Iterative Multiuser Decoding

In order to explain the serial receiver algorithm, let us consider the user with number \( k \). The main parts of the decision chain for the kth user in the mth iteration are depicted in Fig. 2.

First, the interference reduced signal \( y_k'[n] = y[n] - S'[n]x_k[n] \) is obtained using previous estimates on the users’ channel symbols which are arranged in \( \hat{x}_k[n-1] = (\ldots, \hat{x}_k[n-1], \hat{x}_k, \hat{x}_k[n+1], \ldots) \). Here, \( \hat{x}_k[n] \) denotes the soft/hard decision on the kth user’s channel symbol made in the mth iteration. The initialization \( \hat{x}_k'[0] = \mathbb{E}[x_1[n]] \) is applied. The soft decision employed is derived as conditional expectation \( \hat{x}_k[n] = \mathbb{E}[x_k[n]|p_k[n,c]|] \), where \( p_k[n,c] \) denotes the probability information provided by the kth user’s MAP-SSE decoder for channel symbol \( x_k[n] \) after the lth decoding loop.

The signal \( \hat{y}_k'[n] \) is fed into an unbiased residual interference suppression filter designed according to the MMSE-criterion

\[
\begin{align*}
\left( w_k[n] \right) = & \frac{\sigma_n^2}{\sigma_n^2 + \mathbb{E}[n_k[n]]^2} \left( \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \right) \left( \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \right) \left( \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \right)^{-1} \left( \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \right)^{-1} \hat{y}_k'[n].
\end{align*}
\]

with \( \sigma_n^2 = \mathbb{E}[x_k[n]]^2 \). The exact solution to \( P_k[n-1] = \mathbb{E}[\hat{x}_k[n] - \hat{x}_k[n-1]|x_k[n], p_k[n,c]|] \) requires to solve the corresponding value \( p_k[n,c] \) of \( \hat{x}_k[n-1] \). Hence, the inherent bias \( w_k[n] \) of an MMSE-filter output is obtained as \( w_k[n] = \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \left( \sigma_n^2 + \mathbb{E}[n_k[n]]^2 \right)^{-1} \hat{y}_k'[n]. \)

With this, the filter output \( f_k[n] = (w_k[n], \ldots) \) can be written as \( f_k[n] = \hat{x}_k[n] + \hat{y}_k[n] \). In the following, the interference \( n_k[n] \) modeled as zero mean complex Gaussian random variable \( [n] \). Assuming that \( w_k[n] \) is the correct MMSE-filter the noise variance is obtained as \( \sigma_n^2 = \mathbb{E}[n_k[n]]^2 \). Thus, the symbol probabilities \( p_k[n,c] = \mathbb{E}[x_k[n]|x_k[n] = X_k], \forall X_k \in X_k \), at the output of the interference suppression filter \( w_k[n] \) are solved as

\[
\begin{align*}
p_k[n,c] = \frac{\exp\left( -\frac{1}{\sigma_n^2} \left| X_k \right|^2 \right)}{\sum_{X_k \in X_k} \exp\left( -\frac{1}{\sigma_n^2} \left| X_k \right|^2 \right)}.
\end{align*}
\]

Based on these probabilities as input, the MAP-SSE algorithm delivers probabilities \( p_k[n,c], \forall \mu \), for the kth user’s data symbols as well as probabilities \( p_k[n,c] = \mathbb{E}[x_k[n]|x_k[n] = X_k], \forall X_k \in X_k \), for the channel symbols required for the next iteration.
Equipped with the probabilities for the channel symbols the soft estimate for the next iteration cycle is given by $x_{k+1}[u] = \sum_{i,v, \in x_k} x_{i,v} P_{i,v}[u] \cdot x_{k+1}[u]$. \(\forall u\), exploiting the reliability information provided by the decoder. With these new estimates the interference suppression filter for the succeeding user \(k+1\) can be calculated.

This serial interference suppression and decoding procedure is carried out for iteration \(m = 1, 2, \ldots\) for all users and all transmission intervals.

IV. Analytic Solution of Receiver Performance

In this part of the work we focus on the problem to obtain an analytical solution enabling the theoretical prediction of the bit error ratio (BER) resulting by application of the iterative receiver described above. This derivation is done under the supposition that the spreading factor \(N\) goes to infinity while the load \(\beta = K/N\) is constant. This implies \(K \to \infty\), too. The suitability of this assumption, which is made here for the theoretical approach, will be shown in the next section. There, the analytical results are compared with real systems using a spreading factor of 64.

The basis of our derivation is the asymptotic eigenvalue distribution of \(N \times N\) random covariance matrices in the limit \(N \to \infty\) given in [11] which was used in [7] to study the performance of a linear MMSE-filter for synchronous uncoded transmission over a single-path channel. Here, in contrast to [7], frequency selective fading, the application of serial successive cancellation based on soft outputs of the decoder as well as different iteration cycles must be taken into account.

To explain the calculation, let us again consider user \(k\) and iteration \(m\).

First, the assumed instantaneous signal to noise ratio SNR\(_{k,m}[u]\) at the output of the interference suppression filter \(w_k[u]\) is solved analytically. With the model \(f_k[u] = x_k[u] + n_k[u]\), the desired value SNR\(_{k,m}[u] = \sigma^2_k / \sigma_n^2 [m]\), reads

$$
\text{SNR}_{k,m}[u] = \sigma^2_k s_k^H[u] \left( \sum_{\alpha < k} s_{\alpha}[u] \sigma^2_{\alpha,m}[u] (s_{\alpha}[u])^H \right)^{-1}
$$

\[4\]

Here, \(\sigma^2_k[u] = E\{|x_k[u] - \hat{x}_k[u]|^2|x,c,u\}\) denotes the expected interference power of user \(k\) remaining after soft interference cancellation in iteration \(i\).

In order to eliminate the dependence of SNR\(_{k,m}[u]\) on the users' spreading sequences \(s_k[u], \forall k\), the actual transmission model given in Eq. (1) has to be replaced by an equivalent one. So, neglecting the first \(L - 1\) as well as last \(L - 1\) samples of each effective spreading sequence (being possible for \(N \gg L\) [9]) all remaining elements of the \(k\)th user's effective spreading sequence can be modeled as \(s_{\alpha}[u] = \hat{s}_{\alpha}[u] / \sqrt{\sum_{i=0}^{L-1} |\hat{h}_{\alpha,i}[u]|^2}, \forall k\). With the assumption that the complex variables \(\hat{s}_{\alpha}[u]\) are zero mean Gaussian random variables with variance \(1/N\) we found that the distribution of SNR\(_{k,m}[u]\) is not changed if \(N \gg 1\) (for conditions see [11, 12, 9]). In fact, each user's effective spreading sequence is treated as product \(\hat{s}_k[u] / \sqrt{\sum_{i=0}^{L-1} |h_{k,i}[u]|^2} \triangleq \hat{s}_k[u]a_k[u]\). Thus, our transmission model is now

$$
y_k[u] = \hat{S}_k[u]A[u]a_k[u] + n_k[u],
$$

\[5\]

where \(\hat{S}_k[u] = (\hat{s}_k[u], \ldots, \hat{s}_k[u])\) and \(A[u] = \text{diag}(a_1[u], \ldots, a_K[u])\) stand for the users' spreading sequences and fading amplitudes, respectively. Further, supposing that for the number of users in the system holds \(K \gg 1\), each user can be assigned an amplitude \(a_k[u]\), which is drawn randomly from the chi-square distribution \(f_{\alpha,1}(\alpha) = \frac{\alpha}{2} \Gamma(\frac{\alpha}{2})^{-1} 2^{\frac{\alpha}{2}} - 1 \gamma^{-\frac{\alpha}{2}}(\alpha/\gamma)\). Here \(U(\alpha)\) denotes the unit step function. Note, like for the design of the actual receiver, the dependencies between different time slots and users resulting from iterative decoding are neglected in the following. So, adapting Eq. (4) to the equivalent transmission model yields

$$
\text{SNR}_{k,m}[u] = \sigma^2_k a_k^2[u] \hat{s}_k^H[u] \left( \sum_{\alpha < k} \hat{s}_{\alpha}[u]\sigma^2_{\alpha,m}[u] (\hat{s}_{\alpha}[u])^H \right)^{-1}
$$

\[6\]

For the following it turns out to be convenient to get rid of the dependence on \(a_k[u]\). To achieve this, the normalized signal to noise ratio SNR\(_{k,m}[u]/\sigma^2_k[u]\) is defined. Based on the presuppositions made above and keeping the load \(\beta\) constant the signal to noise ratio SNR\(_{k,m}[u]\) converges in the limit \(N \to \infty\) in probability to

$$
\text{SNR}_{k,m}[u] 1^-N \left( \sum_{\alpha < k} \sigma^2_{\alpha,m} + \text{SNR}_{k,m}[u]\sigma^2_{\alpha,m}[u]a^2_{\alpha}[u] \right)^{-1}
$$

\[7\]

With the assumption that \(K/N\) is fixed for a rising spreading factor \(N\), the limit of the normalized signal to noise ratio is equal for all time slots regardless of the actual channel state and actual soft decision of a single user. This is a consequence of the huge multuser interference diversity for \(K \to \infty\). Hence, the time index \(u\) can be dropped in Eq. (7), where SNR\(_{k,m}[u]\) is solved iteratively from . In order to proceed, we regard that post-multiplication of the MMSE-filter output \(f_k[u] = x_k[u] + n_k[u]\) by \(a_k[u]\) leads on the one hand to

$$
\tilde{f}_k[u] \triangleq a_k[u]x_k[u] + \lambda_k[u],
$$

\[8\]

while on the other hand the signal to noise ratio remains unchanged SNR\(_{k,m}[u]\) = \(a_k^2[u] \sigma^2_k[u]/\sigma^2_n[u]\). Rewriting the last equation we get \(\tilde{f}_k[u] = \sigma^2_k[u]/\text{SNR}_{k,m}[u]\), and it reveals that the variance \(\sigma^2_{\alpha,m}[u]\) of the zero mean Gaussian noise \(\lambda_k[u]\) is equal for all time slots. At this point, we would like to emphasize that by multiplication of \(f_k[u]\) by the known scalar \(a_k[u]\) the decoder's soft output for user \(k\) is not changed.

Of course, this multiplication is considered appropriately in calculation of the decoders soft inputs. So, employing the model described by Eq. (8) and equipped with the variance \(\sigma^2_{\alpha,m}[u]\) it is possible to solve the resulting Bit Error Ratio BER\(_{k,m} = f_{\text{BER}}(\sigma^2_k[u], \sigma^2_{\alpha,m}, L)\) of user \(k\) in iteration \(m\). The function \(f_{\text{BER}}(\cdot)\) describes the so-called single user bound (SUB). It is obtained by measuring the BER achievable by a single user transmitting with power \(\sigma^2_k[u]\) over a channel with
weights drawn according to \( f_{\alpha,\xi}(\alpha) \) and being interfered by additive Gaussian noise with variance \( \sigma_{\Delta}^2[m] \) for a specific code employed. Besides \( \text{BER}_m \), the distribution of the soft decisions \( \mathbf{x}_n^T[m] \) provided by the MAP–SSE is required. This distribution is completely determined by the probability density function \( f_{\beta}(\sigma_{\Delta}^2, \alpha, \mathbf{x}_n^T[m]) (\alpha) \) relying on the signal and noise power as well as the channel state. Like \( f_{\text{BER}}(\cdot) \), it can be derived by measurement of the decoder's soft outputs for various signal and noise variances and fading channels. Equipped with \( \sigma_{\Delta}^2, \alpha, \mathbf{x}_n^T[m] \), \( \mathbf{z}_n^T[m] \) can be randomly chosen from \( f_{\beta}(\sigma_{\Delta}^2, \alpha, \mathbf{x}_n^T[m]) (\alpha) \) and the power \( \sigma_{\Delta}^2[m] \) remaining after cancellation is solved as \( \sigma_{\Delta}^2[m] = \mathbb{E} \{ x_n^T[m] – \mathbf{z}_n^T[m] \}^2 \).

Another possibility to obtain \( \sigma_{\Delta}^2[m] \) would be to solve the expectation which reads

\[
\sigma_{\Delta}^2[m] = \mathbb{E} \{ x_n^T[m] – \mathbf{z}_n^T[m] \}^2 \mid \mathbb{P}_m \cdot \mathbb{C}[m] \} = \sigma_{\Delta}^2[m] – \int_0^\infty \mathbb{E} \{ x_n^T[m] – \mathbf{z}_n^T[m] \} (\alpha) \text{d} \alpha. \quad (9)
\]

(10)

However, it is found that both ways lead to identical analytical results. This seems to be a consequence of the large number of users assumed, too. In this way, the calculation is carried out for all users and iterations. Finally, the average Bit Error Ratio after the \( \nu \)th iteration is obtained as

\[
\text{BER}^m = \frac{1}{K} \sum_{k=1}^{K} \text{BER}^m_k \xrightarrow{K \to \infty} \mathbb{E} \{ \text{BER}_m \}. \quad (11)
\]

V. Results

In order to show the tight agreement of the simulated BER as well as analytical solution some numerical results are presented. In the studies of a rate \( R_c = 1/2 \) convolutional code with 64 states and randomly chosen interleavers for all users are applied. Further, a QPSK signal constellation with Gray labeling is employed and equal transmit energy users are supposed so that \( \sigma_{\Delta}^2 = \sigma_{\Delta}^2, \forall k \).

In Figs. 3, 4 and 5 the measured BER versus the required energy per bit to noise ratio \( E_b/N_0 \) are depicted for \( \beta = 1, 1.5, 2 \), respectively. Here, \( N_0 \) denotes the one-sided power spectral density of the continuous-time additive white Gaussian channel noise. The maximum number of iterations carried out is 2 for \( \beta = 1 \) and 4 for \( \beta = 1.5, 2 \). In addition, the curves resulting from the analytical approach are included. Further, \( N = 64 \) and \( L = 1, 3, 5 \) are chosen so that the condition \( N \gg L \) is well satisfied.

Finally, for comparison the single user bounds belonging to the BER of a single user transmitting over frequency selective channels with \( L \) paths and performing ideal maximum ratio combing at the receiver without any distortion due to intersymbol or interpath interference are depicted.

It can be seen from the figures that regardless of the load, the number of paths as well as the iteration cycle, the simulated and analytic curves for the Bit Error Ratio coincide almost completely. In particular the convergence of the multiuser receiver’s performance to the single user bound can be anticipated exactly by the analytic solution. In addition, the assumptions made in the design of the receiver turn out to be reasonable. So, in the theoretical approach the correlations between soft decisions of different users arising in course of the iterations have been ignored completely. From the good accordance of the analytical and simulated results it can be inferred that this holds for the actual receiver scheme to a large degree, too. Of course, these correlations depend heavily on the interleaver size chosen, as it is well-known for Turbo-Codes, and the system load \( \beta \). Hence, we reckon that the small gaps appearing for \( \beta = 2 \) and \( 10 \log_{10} (E_b/N_0) = 5 \text{ dB} \) and \( L = 1 \) as well as \( 10 \log_{10} (E_b/N_0) = 4 \text{ dB} \) and \( L = 3 \) could be closed by choosing a larger interleaver.

On the other hand, our investigations showed that the Gaussian approximation is well satisfied already for \( N = 64 \). So, larger spreading factors did not provide a performance improvement. Moreover, it should be noted, that the derivation presented in Section IV is valid also for other randomly chosen spreading sequences as long as the chips are independent (see [11]). Thus, for sufficiently large spreading factor the performance of the system won’t be affected by the chip distribution.

Next, the plots underline the advantages offered by
confirmed that orthogonal multiple access schemes indeed can be outperformed in terms of spectral efficiency. Finally, having restricted in this paper to the MMSE receiver for sake of clarity, we can show that in the same way the performance of other iterative multiuser receivers for frequency selective fading channels with and without perfect channel state information can be predicted, too.

REFERENCES


Figure 5: BER vs. $10 \log_{10}(E_b/N_0)$ for $\beta = 2$ and $L = 1$ (●), $L = 3$ (•) and 4th (○) iteration for simulated results, after 1st (□) and 4th (△) iteration for analytical results and corresponding SUB (○).