Abstract

The class of interleavers is identified, which possesses the following property: both Turbo Code component scramblers are terminated in the same state, independent of the encoder information word. It is proven that the presented class contains all interleavers with the above property.

Introduction: Turbo Codes [4] represent the most power-efficient binary channel codes of the present time. Their encoder consists of three parallel branches: one systematic branch and two (recursive) scramblers (in the following denoted by scr1 and scr2, respectively), both of which are joined by an interleaver. In the article we always assume identical scramblers of memory $\nu$.

Scrambler properties: Throughout the article, we use the following terminology. Variables $\bullet$ with a superscript index $\bullet^{(1)}$ and $\bullet^{(2)}$ are associated with scr1 and scr2, respectively. The input of length $K$ to the scramblers is denoted by the vector $u^{(j)} = (u_0^{(j)}; u_1^{(j)}; \ldots; u_{K-1}^{(j)}), j \in \{1; 2\}$ with $u_i^{(j)} \in \text{GF}(2) = \{0; 1\}$. The interleaver function $u^{(2)} = \pi(u^{(1)})$ performs transpositions $i \rightarrow \pi_i$ of the input bit positions $i = 0..K-1$ of scr1 to those of scr2, i.e. $u_i^{(2)} = u_i^{(1)}$. Let the vectors of $\nu$ binary elements $S^{(1)}_K = (S^{(1)}_{K; 1}; S^{(1)}_{K; 2}; \ldots; S^{(1)}_{K; \nu-1})$ and $S^{(2)}_K = (S^{(2)}_{K; 1}; S^{(2)}_{K; 2}; \ldots; S^{(2)}_{K; \nu-1})$ with $S^{(j)}_{K; l} \in \text{GF}(2)$ denote the terminating state of scr1 and scr2 respectively, i.e. the state of the scrambler shift register after the $K$ input bits of $u^{(1)}$ and $u^{(2)}$ are fed into these scramblers. In the following, all sum (+) and multiplication (\cdot) operations are carried out in $\text{GF}(2)$.

The aim of this article is to find and present all Turbo Code interleavers having the property that the trellises of both scramblers terminate in identical (but otherwise arbitrary) states $S^{(1)}_K = S^{(2)}_K$ after the input of the $K$-bit encoder information word $u = u^{(1)}$, independent of the information word $u$. In the following, this property is referred to as the “double terminating” property. Note that by terminating scr1 in e.g. the zero-state by choosing an appropriate information word $u$, also scr2 is terminated in this particular state, when the Turbo Code possesses the double terminating property.
We examine the effect of feeding arbitrary input vectors to scrambler scr1 only. The function \( f : \mathbf{u}^{(1)} \rightarrow S_K^{(1)} \) represents the mapping from scrambler input vectors to terminating states. Since the scrambler is a linear device, this function is linear:

\[
f(a \cdot \mathbf{u}_1 + b \cdot \mathbf{u}_2) = a \cdot f(\mathbf{u}_1) + b \cdot f(\mathbf{u}_2) \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in [\text{GF}(2)]^K, a, b \in \text{GF}(2).
\]

Since the terminating state \( S_K^{(1)} \) can take on \( 2^p \) different possible values \( s_j, j = 0..2^p - 1 \), the mapping \( f \) subdivides all possible input vectors \( \mathbf{u}^{(1)} \) into \( 2^p \) disjoint sets \( \mathcal{U}_j, j = 0; 1; \ldots; 2^p - 1 \). All vectors \( \mathbf{u}^{(1)} \) belonging to one set \( \mathcal{U}_j \) are mapped to the same terminating state \( S_K^{(1)} = s_j \); if any two vectors belong to different sets, their associated terminating states are hence different.

When a scrambler is excited by a single one at its input, it periodically cycles through \( p \) different shift register states \( s_0; s_1; \ldots; s_{2^p-1} \) different from the zero-state, where \( p \) only depends on the realisation of the scrambler and is always a factor of \( 2^p - 1 \). Note that for this notation \( s_0 \) does not represent the zero-state. Let \( \mathbf{e}_i = (e_i;0 = 0; e_i;1 = 0; \ldots; e_i;K-1 = 1), i = 0..K - 1 \) denote the \( K \) unity vectors. Since there are only \( p \) possible terminating states \( S_K^{(1)} \in \{s_0; s_1; \ldots; s_{2^p-1}\} \) for \( \mathbf{u}^{(1)} = \mathbf{e}_i \), the \( K \) unity vectors are subdivided into the \( p \) disjoint sets \( \mathcal{U}_0; \mathcal{U}_1; \ldots; \mathcal{U}_{2^p-1} \). Because of the periodicity of the scrambler, we have the following subdivision \( \mathbf{e}_i \in \mathcal{U}_{i \mod p} \), i.e. \( f(\mathbf{e}_i) = s_{i \mod p} \).

For more information about this subject, cf. [5],[7],[8].

Every input vector \( \mathbf{u}^{(1)} \) of scr1 can be represented as a linear combination of unity vectors:

\[
\mathbf{u}^{(1)} = \sum_{i=1}^{K} u_i^{(1)} \cdot \mathbf{e}_i, \tag{1}
\]

The associated terminating state for \( \mathbf{u}^{(1)} \) then is:

\[
S_K^{(1)} = f(\mathbf{u}^{(1)}) = \sum_{i=1}^{K} u_i^{(1)} \cdot f(\mathbf{e}_i). \tag{2}
\]

**Proposition:** For a given interleaver length \( K \) and identical component scramblers scr1 and scr2, a Turbo Code interleaver \( \pi \) possesses the double terminating property, **if and only if** \( \pi_i \mod p = i \mod p \forall i \in 0..K - 1 \).

In other words, the interleaver has to permute all unity vectors \( e_i \) within their respective sets, i.e. if \( e_i \in \mathcal{U}_j \) then also \( \pi(e_i) = e_{\pi_i} \in \mathcal{U}_j \forall i = 0..K - 1 \), because \( j = i \mod p = \pi_i \mod p \) (see above).

**Proof:** (if) Given: \( \pi_i \mod p = i \mod p \forall i \in 0..K - 1 \). For a given \( \mathbf{u}^{(1)} \) (cf. Eq. (1)), the terminating state \( S_K^{(1)} \) is given in Eq. (2). The terminating state \( S_K^{(2)} \) of the associated

\[
\mathbf{u}^{(2)} = \pi(\mathbf{u}^{(1)}) = \sum_{i=1}^{K} u_i^{(2)} \cdot \mathbf{e}_{\pi_i},
\]

is equal to

\[
S_K^{(2)} = f(\mathbf{u}^{(2)}) = \sum_{i=1}^{K} u_i^{(2)} \cdot f(\mathbf{e}_{\pi_i}).
\]

Hence, the associated terminating state for \( \pi(\mathbf{u}^{(1)}) \) is the same as the terminating state for \( \mathbf{u}^{(1)} \) and the interleaver possesses the double terminating property.
\[ S_K^{(2)} = f(u^{(2)}) = \sum_{i=1}^{K} u_i^{(1)} \cdot f(e_i) \]
\[ = \sum_{i=1}^{K} u_i^{(1)} \cdot f(e_i) = S_K^{(1)}. \]

(only if) Let \( \pi \) be an interleaver with the double terminating property. Assume that there exists a transposition \( \pi_m \mod p \neq m \mod p \). Then for \( u^{(1)} = e_m \), we obtain \( S_K^{(1)} = s_{m \mod p} \) (see above), but for \( u^{(2)} = e_{\pi_m} \) we have \( S_K^{(2)} = s_{\pi_m \mod p} \neq S_K^{(1)} \). Thus, there is a contradiction to the double terminating property and the assumption must be false.

We denote the set of interleavers with the double terminating property as the “class of double terminating interleavers”. For the special case of scramblers with a period \( p = \nu + 1 \), we obtain the “simile” interleavers of [2]. One sub-type of these are the “helical” interleavers presented in [2].

There exist rectangular double terminating interleavers. Generally, for rectangular \( m \times n \)-interleavers, the transpositions are defined as follows: \( i \rightarrow \pi_i = (i \mod n) \cdot m + i \div n \). For rectangular interleavers belonging to the class of double terminating interleavers, we have hence the necessary and sufficient condition:

\[ i \mod p \equiv [(i \mod n) \cdot m + i \div n] \mod p, \quad \forall i = 0..K - 1. \]

If we set \( i = 1 \) we obtain \( 1 = m \mod p \). When we set \( i = n \), we have \( n \mod p = 1 \). Hence all double terminating rectangular interleavers have \( m = j \cdot p + 1 \) rows and \( n = l \cdot p + 1 \) columns, where \( j \) and \( l \) are positive integers.

**Example:** If we choose the scrambler’s memory \( \nu = 2 \) and period \( p = 3 \), the \( 13 \times 13 \)-rectangular interleaver is double terminating. The associated Turbo Codes have information words of length 169 bits, where 2 bits are needed to terminate both scrambler trellises in (not necessarily) the zero-state. Moreover, this interleaver has the “even-odd” property of [1], when puncturing is applied to increase the data rate to 1/2, and it is quadratic, which is useful for decorrelating the component decoder’s extrinsic probabilities.

**Simulation Results:** We compared the \( 13 \times 13 \)-rectangular interleaver to some other interleavers of similar length. The resulting word error rate (WER) is shown in Figure 1. All curves are obtained using a (punctured) Turbo Code of rate 1/2 with component scramblers of memory \( \nu = 2 \). In all cases, a termination of both component scrambler trellises was used, which requires additional encoding effort for interleavers without the double terminating property. The results were obtained
After 10 iterations of the decoder, there were 0 erroneous codewords for every simulated signal to-noise ratio (SNR). For low WERs, the 13 × 13-interleaver performs better than all the other considered interleavers, which is probably due to an increased minimum distance. The other interleavers were a uniform interleaver [3] of length $K = 169$, which was obtained by averaging the WER over randomly chosen interleavers, the 12 × 16 rectangular interleaver (length $K = 192$) of [6] and a 7 × 24 helical interleaver ($K = 168$), cf. [2]. As regards to the bit error rate (BER), again the 13 × 13-interleaver performs best with $2.7 \cdot 10^{-5}$ at an SNR of $10 \log_{10} E_b/N_0 = 3$ dB.

**Final Remark:** In contrast to conventional interleaver design, the purpose of double terminating interleavers is not the maximization of the code's minimum distance, but to achieve double trellis-termination for the component decoders. The interleaver design algorithm of [7] tries to permute every $u^{(1)}$ of weight 2 associated with the zero-terminating state to a $u^{(2)} = \pi(u^{(1)})$ belonging to a different set than $u^{(1)}$ ([7],[8] show that this is possible only for a fraction of all $u^{(1)}$). On the other hand, double terminating interleavers permute all $u^{(1)}$, such that $u^{(1)}$ and $u^{(2)}$ always belong to identical sets. Double-terminating interleavers can still have a good BER performance, since at small lengths $K$, the minimum distance is not only caused by $u^{(1)}$ of weight 2, but also by higher input weights. Hence by permuting $u^{(1)}$ within its respective set, the BER performance is not necessarily deteriorated. On the other hand, terminating both component scrambler trellises becomes very simple.

**References**


Legend for the figure:

- 13 × 13 rect.
- - - uniform $K = 169$
- .... 12 × 16 rect.
- - - - 7 × 24 helical