Abstract—An overview on equalization schemes applicable for point-to-point communication over MIMO ISI channels is given, i.e., channels which suffer from both, multiuser interference between the data streams transmitted in parallel and intersymbol interference due to the dispersive (frequency-selective) nature of the channel. Spatial, temporal and combined spatial/temporal equalization strategies for dealing with these types of interferences are discussed. In particular, linear and decision-feedback equalization, and equalization based on singular value/eigenvector decomposition and lattice basis reduction, respectively, are treated. The underlying mathematical principle, utilized in these schemes, is stated in each case. Via numerical simulations, the performance of selected equalization strategies is compared.

I. INTRODUCTION

It is now widely accepted that for achieving very high spectral efficiencies—and in turn data rates—the use of antenna arrays in transmitter and receiver, hence resorting to multiple-input/multiple-output (MIMO) communication, is the most promising strategy.

Over the last years, mainly flat-fading MIMO channels have been studied. But when increasing data rates even more by using larger and larger bandwidths, the frequency selectivity of the channels becomes more and more pronounced. Then, in addition to multiuser interference (MUI) between the data streams transmitted in parallel over the antennas, intersymbol interference (ISI) due to channel dispersion occurs. Hence, next-generation high-rate transmission schemes have to deal with such MIMO ISI channels.

The aim of the present paper is to give an overview on equalization schemes applicable for communication over MIMO ISI channels. For that, we first treat both phenomena, MUI, i.e., spatial interference, and ISI on single-input/single-output (SISO) channels, i.e., temporal interference, separately. Spatial and temporal equalization strategies for dealing with these types of interferences are discussed. In particular, linear equalization, decision-feedback equalization, equalization based on singular value/eigenvector decomposition, and lattice basis reduction are treated, and the respective mathematical principle, on which the schemes are based, is stated. Then, for each strategy, the combination to combined spatial/temporal equalization is presented.

Please note, throughout the paper we restrict ourself to point-to-point communication, i.e., transmitter and receiver, respectively, have full access to all antenna elements [4]. This is in contrast to multipoint-to-point (typically the uplink) or point-to-multipoint (typically the downlink) scenarios.

However, most of the presented equalization strategies are applicable in either of the mentioned situations, dependent on whether joint transmitter or joint receiver side processing is required. Moreover, maximum data rate is desired. Hence, no methods for increasing diversity at the price of data rate—so-called space-time codes, e.g. [1]—are considered.

Section II gives the general MIMO ISI channel model. In Section III, the equalization strategies are explained and compared. Results from numerical simulations are given in Section IV, and Section V concludes the paper.

II. CHANNEL MODEL

Point-to-point transmission using antenna array with \(N_T\) transmit and \(N_R\) receive antennas is considered. Over the antennas, mutually independent, i.i.d. sequences \(\{a_{\mu}[k]\}\) of \(T\)-spaced data symbols \(a_{\mu}[k], k \in \mathbb{Z}, \mu = 1, 2, \ldots, N_T\), taken from (possibly different) signal constellations \(A_{\mu}\) (variance normalized to \(\sigma_{\mu}^2 \triangleq E\{|a_{\mu}[k]|^2\}, \forall \mu\), should be communicated.\(^2\) The data symbols are combined into the vector \(\mathbf{a}[k] \triangleq [a_1[k], \ldots, a_{N_T}[k]]^T\).

The actual transmission channel consists of three parts. First, at the transmitter, pulse shaping is performed to obtain a continuous-time transmit signal. Then the signal is modulated to the radio frequency band. Second, the radio signal is transmitted over the antenna array and propagates through the channel. In high-rate systems using large bandwidths, the frequency selectivity of the channel becomes important. Hence, the individual channels from each transmit to each receive antenna have to be characterized by (continuous-time) impulse responses rather than single (flat-fading) scaling gains. At the receive antennas, the superpositions of all linearly distorted transmit signals and the unavoidable additional noise are present. Third, the radio frequency receive signals are demodulated and passed through a bank of receive filters, at the output of which \(T\)-spaced sampling is performed. Note, in [8] it has been proved that the optimum (with respect to a wide range of criteria) receiver front-end consists of a bank of matched filter for the cascade of pulse shaping filters and continuous-time channel. All further processing steps may then

\(^1\)In case of transmitter side preprocessing, the actual transmit symbols, denoted as \(x_{\mu}[k]\), are calculated from the mutually independent, i.i.d. data symbols \(a_{\mu}[k]\), and hence may exhibit correlations.

\(^2\)Notation: Matrices are printed in boldface uppercase letters, column vectors in boldface lowercase letters, and scalars in regular lowercase letters. \(\mathbf{A}^T\): transposition; \(\mathbf{A}^{-1}\): inverse of the Hermitian transpose of a square matrix \(\mathbf{A}\). \(\mathbf{I}\): Identity matrix. \(\mathbf{z}^{-*}\) abbreviates \((z^{-1})^*\). \(E\{\cdot\}\): expectation. \(\mathcal{F}_\kappa\{\cdot\}\): Fourier transform with respect to \(\kappa\).
be performed by a discrete-time filter, succeeding the $T$-spaced sampling.

Combining the above three parts, an end-to-end $T$-spaced discrete-time channel model (given in the equivalent complex baseband domain [30], [27]) between vector of data symbols $\mathbf{a}[k] \triangleq [a_1[k], \ldots, a_{N_R}[k]]^\mathsf{T}$ and vector of received symbols $\mathbf{y}[k] \triangleq [y_1[k], \ldots, y_{N_T}[k]]^\mathsf{T}$ can be established. Thereby we assume that the receiver (in particular the discrete-time filter part) is adjusted such that the multiple whitened matched filter [8] is implemented. This receive filter (i) provides a lossless transition from the continuous-time to the discrete-time model (even though sampling theorem is usually not met), and (ii) it is the optimal starting point for a wide class of subsequent equalization strategies [12]. Assuming additionally that the channel is (approximately) constant over each transmission burst, each individual channel from each transmit to each receive antenna $3$ is characterized by a time-invariant, causal impulse response $\langle h_{\nu \mu}[k] \rangle$, $k \geq 0$. Due to the multiple whitened matched filter, the additive discrete-time Gaussian noise is spatially and temporally white with variance $\sigma_n^2 = \frac{1}{N_T}$, i.e., the correlation matrix of the additive noise vector $\mathbf{n}[k]$ reads $\mathcal{F}_\kappa \{ \mathbb{E} \{ \mathbf{n}[k + \kappa] \mathbf{n}^\mathsf{H}\} \} = \mathcal{F}_\kappa(\mathbf{e}^{2\pi j T}) = \sigma_n^2 \mathbf{I}$.

The taps of the individual impulse responses are advantageously combined into $N_R \times N_T$ tap matrices

$$
\mathbf{H}_k \triangleq \begin{bmatrix}
    h_{11}[k] & \cdots & h_{1N_R}[k] \\
    \vdots & \ddots & \vdots \\
    h_{N_R1}[k] & \cdots & h_{N_RN_T}[k]
\end{bmatrix},
$$

(1)

This channel is completely characterized by the matrix channel transfer function

$$
\mathbf{H}(z) = \sum_{k=0}^{L-1} \mathbf{H}_k \, z^{-k},
$$

(2)

where $L$ is the maximum length of the impulse responses ($\mathbf{H}(z)$ is a matrix polynomial of degree $L - 1$). Thus, the action of the discrete-time linear dispersive vector channel is described by

$$
\mathbf{y}[k] = \sum_{k=0}^{L-1} \mathbf{H}_k \mathbf{a}[k - \kappa] + \mathbf{n}[k].
$$

(3)

Figure 1 visualizes the channel impulse response, which is dependent on the three parameters transmit antenna $\mu$, receive antenna $\nu$, and discrete time index $k$. Transmitting a single unit impulse over antenna $\mu$, the impulse response for the given $\mu, \nu = 1, \ldots, N_R$, and $k \geq 0$ is observed which, in the visualization, lies in the $\nu$-$k$ plane.

The above channel model contains two special cases: First, if all individual channel impulse responses are of length 1, only $\mathbf{H}_0$ has non-zero entries and the common flat-fading MIMO channel model results.

Second, for $N_T = N_R = 1$ a SISO ISI channel is present. $5$ Both extreme cases are visualized in Figure 1, too. In the first case, only spatial interference is present; the data symbols transmitted in parallel at the same time interfere (also called multiuser interference). For reliable detection of the transmitted symbols, equalization with respect to the spatial dimension has to be performed. In the second scenario, temporal interference (intersymbol interference) is active; data symbols of a single user transmitted in sequence interfere. Here, equalization with respect to time is required. In the general setting, where both spatial and temporal interferences occur, joint spatial/temporal equalization has to be implemented.

### III. Characterization of Equalization Strategies

In this section, equalization strategies for spatial and temporal equalization are discussed. We thereby first concentrate on pure spatial and pure temporal equalization. Then, combined spatial/temporal equalization strategies are presented. In each case, perfect channel knowledge at the receiver (if specified at the transmitter, too) is assumed.

Please note, in this paper we restrict ourselves to symbol-by-symbol detection schemes. We hence do not consider maximum-likelihood detection (MLD, spatial) [27], maximum-likelihood sequence estimation (MLSE, temporal) [15], or joint space/time MLSE, which is optimum for MIMO ISI channels [8]. Such schemes are in general much too complex. Assuming a MIMO ISI channel with parameters $N_T$, $N_R$, and $L$, and cardinalities $M_{\mu} = |A_{\mu}|$ of the signal constellations, a Viterbi-algorithm-based MLSE scheme would have $\prod_{\mu=1}^{N_R} M_{\mu}^{(L-1)}$ states and $\prod_{\mu=1}^{N_R} M_{\mu}$ branches leaving and entering each state. Even for moderate values (e.g., $N_T = 3, L = 5$, and $M_{\mu} = 16$), the number of states ($16^4 \cdot 16^8 = 2.8 \cdot 10^{14}$) and branches ($16^3 = 4096$; i.e., $16^{12} \cdot 16^3/(3 \cdot 4) = 2^{58}/3$ trellis branches would have to be visited per decoded bit of information) become prohibitively large. We hence concentrate on equalization for sub-optimal, but low-complex—and for some cases even close to optimal—symbol-by-symbol detection schemes.

#### A. Linear Equalization

The most obvious strategy for separating or equalization the data streams transmitted in parallel is linear equalization $6$ at the receiver side.

1) Spatial Equalization: In case of flat-fading MIMO channels, the decision vector $\mathbf{r}$—which in uncoded transmission is passed to the threshold device to produce the estimate $\hat{a}$—is generated by $\mathbf{r} = \mathbf{H}_0^{-1} \mathbf{y}$. Here, $\mathbf{H}_0^{-1}$ denotes the left (pseudo) inverse of the channel matrix $\mathbf{H}$ (existence assumed). This equalization strategy requires $N_R \geq N_T$, i.e., the number of receive antennas has to be at least as large as the number of data streams transmitted in parallel. Moreover, using this approach, interference of the data streams are completely suppressed; it is hence called zero-forcing (ZF) linear equalization. The receive filter may also be optimized according to the minimum mean-squared error.

2) Linear Detection: In literature the denomination linear detection is also used sometimes. We prefer the term linear equalization, because the processing for equalization is linear. The subsequent detection process, by definition, is always non-linear. The term linear detection hence is an oxymoron.
Fig. 1. Illustration of spatial/temporal impulse response of MIMO ISI channels (left, $N_T = 3$ transmit, $N_R = 4$ receive antennas) and special cases flat-fading MIMO channel (only spatial interference, middle) and single-input/single-output ISI channel (only temporal interference, right). Each box corresponds to a sample of the impulse response between antennas $\mu$ and $\nu$ at time delay $k$. The color may be interpreted as the average power.

Fig. 2. Transmission systems with linear equalization (top) and linear pre-equalization (bottom). Left: spatial equalization for flat-fading MIMO channels; Right: temporal equalization for SISO ISI channels. Vectorial (parallel) signals are drawn with double-line arrows, whereas scalar signal are symbolized with single lines.

(MMSE) criterion, resulting in MMSE linear equalization. Here, instead of $H_H^{-1} = (H_H^H H_H)^{-1} H_H^H$ the filter matrix $(H_H^H H_H + \frac{\sigma^2}{\varsigma^2} I)^{-1} H_H^H$ is used.

Figure 2 sketches the structure of a MIMO system using linear receiver-side equalization.

If the channel matrix $H$ is also available at the transmitter—either by communicating it to the transmitter via a backward channel or in time-division duplex (TDD) systems assuming reciprocity of the channel—the data streams can be separated by means of pre-equalization [2], [20], see Figure 2. Interference of the users at the receiver side can be completely avoided (ZF approach) by feeding the pre-distorted version $\tilde{x} = \varsigma H_1^{-1} \tilde{a}$ instead of the data vector $a$ itself into the channel. For that, the data vector $a$ is filtered by the (pseudo) right inverse $H_1^{-1}$ of the channel matrix $H$. The real constant $\varsigma$ is chosen such that average transmit power is constant for each channel realization and equals that of a direct transmission of the data vector $a$ ($N_R \cdot \sigma_a^2$ in the present case). This constant is compensated by an automatic gain control (AGC, pure scaling) at the receiver. Linear pre-equalization requires $N_T \geq N_R$.

If an MMSE approach is desired, $H_H^{-1} = H_H^H (H_H H_H)^{-1}$ is replaced by the filter matrix $H_H^H (H_H H_H + \frac{\sigma^2}{\varsigma^2} I)^{-1}$ [29], [31].

2) Temporal Equalization: In case of SISO ISI channel, the sequence $\langle r[k] \rangle$ of decision variables is obtained by passing the receive signal $\langle y[k] \rangle$ through a linear filter with transfer function $1/H(z)$. This is only possible if $H(z)$ is strictly minimum phase, i.e., all zeros of $H(z)$ have to lie within the unit circle. In particular, zero-forcing linear equalization does not exist in case of spectral zeros. As above, the ZF approach can be replaced by an optimization according to the MMSE criterion; details can be found in the literature [11].

If the channel transfer function $H(z)$ is known to the transmitter, linear pre-equalization via $\varsigma/H(z)$ is possible. The scalar is again chosen such that a fixed transmit power is guaranteed.

In Figure 2, the SISO ISI transmission system with linear receiver- and transmitter-side equalization is shown, too.

3) Spatial/Temporal Equalization: The linear equalization approach can easily be extended to combined spatial/temporal equalization. Now, a matrix filter, i.e., a bank of linear (time-invariant) filters, with transfer function $H_0^{-1}(z)$ is required. The left inverse $H_0^{-1}(z) = \sum_{k} H_k z^{-k}$ of the matrix polynomial is defined by $H_0^{-1}(z) H(z) = I$, where $I$ is the identity matrix, and can be successively calculated by [9]

$$H_0 = H_0^{-1}, \quad H_k = -\left(\sum_{i=0}^{k-1} H_i H_{k-i}\right) H_0^{-1}.$$ (4)

The same principle can be applied for spatial/temporal linear pre-equalization. Due to non-commutativity of matrix multiplication, here a right inverse $H_0^{-1}(z)$ of the matrix polynomial $H(z)$ is required, which can be calculated similarly as above.

Footnote 3For $N_R > N_T$ the matrix inverse has always to be understood as the (Moore-Penrose) pseudo inverse.
It is well-known that the presented simple linear equalization strategies, either for spatial or temporal equalization, suffer from noise enhancement and hence have poor power efficiency. The same is true for linear pre-equalization if (via scaling of the transmitter side filter) a constant transmit power is forced and noise is scaled by the receiver side AGC. In case of spectral zeros/singular matrices, (ZF) linear equalization can not be implemented in a stable way.

B. Decision-Feedback Equalization and Precoding

To some extent, the disadvantages of linear equalization can be overcome by decision-feedback equalization (DFE). The main idea to improve system performance is to utilize decisions already taken for interference cancellation and noise prediction; a concept known for a very long time, see the references in [26]. This principle of successive processing has its justification in the chain rule of information theory [6] and is known as successive cancellation in the field of multiuser detection, cf. [7].

1) Spatial Equalization: The application of DFE for MIMO systems is known for long time (e.g., [7], [9]), but only one time instant are detected.

Assuming (correct) decisions \( \hat{a}_\mu = a_\mu \) on the data symbols \( a_\mu, \mu = 1, 2, \ldots, N \), are already available, their contribution (interference) to the received symbol \( y \) can be subtracted out, i.e., \( y_\kappa = y - \sum_{\mu=1}^{\kappa-1} h_\mu a_\mu \), where \( h_\mu \) denotes the \( \mu \)-th column of the channel matrix \( H \). Calculated, then, this signal is filtered with a filter vector \( f_\mu \) in order to suppress the interference of not yet detected symbols. Based in the decision variable \( f^H_\mu y_\mu \), an estimate of the data symbols \( a_\mu \) is generated. This procedure is repeated until all symbols at one time instant are detected.

For optimum performance, in general, the data symbols should not be detected in the original order \( a_1, \ldots, a_N \), but in an optimized one [18]. This corresponds to a virtual relabeling of the transmit antennas.

A closer look into the detection algorithm reveals that the resulting transmission system has the structure as shown on top of Figure 3. It consists of a feedforward matrix \( F \), composed of the row vectors \( f_\mu^H \), a feedback matrix \( B \), and a (real) permutation matrix \( P \), which represents the decision order. The task of the matrix \( F \) is to transform the cascade \( FHP \) into lower triangular shape with unit main diagonal (ZF approach).

The interference of already detected symbols is then subtracted by the strictly lower triangular matrix \( B - I \). In the last step, the original order is reestablished via the matrix \( P \).

Given the end-to-end channel matrix \( H \), the required matrices are calculated such that [11], [31]

\[
FHP = B^H \Sigma B.
\]  
\[6\]

i.e., a sorted QR-type factorization has to be performed. Given a criterion of optimality (e.g., maximum signal-to-noise ratio in each detection step), the factorization is unique. Noteworthy, defining a real diagonal matrix \( \Sigma \), the calculation can also be done via a Cholesky factorization [19]

\[
P^H H^H P = B^H \Sigma B.
\]
\[6\]
The feedforward matrix is obtained as \( F = \Sigma^{-1} B^{-H} P^H H^H \).

Of course, the DFE can also be optimized according to the MMSE criterion instead of using ZF DFE. Details can be found, e.g., in [31].

2) Temporal Equalization: Decision-feedback equalization for channels with intersymbol interference is very popular. As for spatial equalization, temporal DFE also consists of a feedforward filter \( F(z) \) and a feedback filter \( B(z) \), which subtracts out the interference caused by already detected symbols \( \hat{a}[k - \kappa], \kappa = 1, 2, \ldots \). The task of the feedforward filter is (i) to shape the discrete-time end-to-end response \( F(z)H(z) \) to be causal and (assume proper scaling) monic (first tap equal to one), and (ii) to produce white noise with minimum variance, effective at the decision device.

In contrast to spatial scenarios, in temporal equalization no optimization of the decision order is possible. Since time has an obvious direction, symbols are always detected in the logical order.\(^8\) However, the end-to-end response \( F(z)H(z) \) may still be optimized. Since the decisions are based on the zeroth tap of the end-to-end impulse response and the trailing coefficients are canceled, the zeroth tap should be as large as possible. This is guaranteed if and only if \( F(z)H(z) \) is minimum phase [11].

Given the channel transfer function \( H(z) \) the optimum feedforward \( F(z) \) and feedback filter \( B(z) \) can be calculated by performing the following spectral factorization [16], [11]

\[
H^*(z^{-\ast})H(z) = B^*(z^{-\ast}) \cdot \sigma \cdot B(z),
\]
\[7\]

where \( \sigma \) is a scaling factor and \( B(z) = 1 + \sum_{k=1}^{\infty} b[k] z^{-k} \) is causal, monic, and minimum-phase. Then, \( B^*(z^{-\ast}) \) (\( z^{-\ast} \) abbreviates \( \ast^{-1} \)) is anti-causal, monic, and maximum-phase. Efficient factorization algorithms exist, e.g., [14]. Having performed the factorization, the feedforward filter calculates to\(^9\)

\[
F(z) = H^*(z^{-\ast})/(\sigma B^*(z^{-\ast})).
\]
\[7\]

3) Spatial/Temporal Equalization: Joint spatial/temporal equalization can efficiently be done by spatial/temporal DFE. The structure of a transmission scheme using DFE is sketched on top of Figure 3. Now, a matrix feedforward filter \( F(z) \), a matrix feedback filter \( B(z) \), and the permutation matrix \( P \) are present.

Here, the task of the feedforward filter is to shape the end-to-end response \( F(z)H(z) \) to exhibit spatial and temporal causality. Symbols transmitted at time instant \( k \) should only interfere into receive symbols at time instances \( \kappa > k \). Moreover,—assuming an optimized order of the antennas—symbols transmitted at the same time instance over antenna \( \mu \) should only interfere into receive symbols for antennas \( m > \mu \). Due to the twofold causality, the symbols can be detected successively in a zig-zag fashion over space and time.

\(^8\)In bursty transmission, when the receiver can operate off-line on the entire received burst, time-reversed processing can be done alternatively.

\(^9\)If \( H(z) \) is already minimum phase and white channel noise is present, \( F(z) \) simplifies to a scalar.
It can be shown that best performance is achieved if (i) $B(z)$ is minimum phase, which for matrix polynomials is defined as $\det(B(z)) \neq 0$, $|z| > 1$, and (ii) an optimized detection order (permutation $P$) is chosen [12].

Given the channel transfer function $H(z)$, the required matrix feedforward and feedback filters are obtained by performing a spectral factorization [11], [12] (note the generalization of (6) and (7))

$$
P^T H^H(z^{-s}) H(z) P = B^H(z^{-s}) \cdot \Sigma \cdot B(z) . \tag{8}
$$

$\Sigma = \text{diag} (\varsigma_1, \ldots, \varsigma_N)$ is a real diagonal matrix, $P$ a permutation matrix, and the matrix polynomial $B(z) = \sum_{k \geq 0} B_k z^{-k}$ is causal and minimum-phase. Moreover, $B_0$ is lower triangular with unit main diagonal. This factorization problem can efficiently be solved by a two-step algorithm [14], which first does the factorization ignoring the permutation and then calculates the optimal $P$. The feedforward matrix filter calculates to $F(z) = \Sigma^{-1} B^{-H} (z^{-s}) P^T H^H(z^{-s})$.

The two main drawbacks of DFE are (i) error propagation, which occurs in the feedback loop, and (ii) the necessity of taking instant decisions, which is prohibitive for the application of channel coding. Only when considering flat-fading MIMO channels, codewords may be transmitted over different antennas, i.e., the data streams of each antenna are encoded separately. The codewords are decoded in temporal (horizontal) direction, and then, using re-encoded code symbols, the interference for the other, yet undecoded, parallel data streams is canceled in spatial (vertical) direction. For ISI and MIMO ISI channels this separation of decoding and equalization in two different dimensions is not possible.

Error propagation and the demand for zero-delay decisions can be overcome by using precoding instead of DFE. Basically, precoding is obtained from DFE by flipping the entire structure, see Figure 3. Now, the data symbols are first permuted by $P^T$, then processed by the nonlinear feedback loop with matrix filter $B(z)$, and finally passed through the feedforward filter $F(z)$. At the receiver side, only scaling (AGC) by $\varsigma^{-1}$ remain. Details on this Tomlinson-Harashima-type precoding can be found, e.g., in [11], [14]. The required matrix filters are obtained by spectral factorization similar—due to non-commutativity of matrix multiplication not identical—to (8).

C. Singular-Value Decomposition and Multicarrier Modulation

Linear (pre-)equalization and DFE/precoding work either entirely on the receiver or the transmitter side. The equalization task can as well be split among transmitter and receiver. Here, approached based on eigenvectors and the diagonalization of matrices are of particular interest.

1) Spatial Equalization: A popular strategy for transmission over flat-fading MIMO channel is based on the singular value decomposition (SVD) of the channel matrix $H$, i.e., it is written as

$$
H = UV^H, \tag{9}
$$

where $U$ and $V$ are unitary matrices ($UU^H = I$) and $\Sigma = \text{diag} (\varsigma_1, \ldots, \varsigma_N)$, with $N = \min \{N_T, N_R \}$, is real diagonal. For each matrix $H$, regardless of its dimension and whether it is singular, such a decomposition is possible [19].

Applying $V$ at the transmitter, i.e., the transmit vector is calculated as $x = Va$ from the data vector, and $U^H$ and $\Sigma^{-1}$ at the receiver, i.e., the decision vector is obtained as $r = \Sigma^{-1} U^H y$, the transmission scheme shown in Figure 4 results. Due to the pre- and post-processing, independent, parallel channels, scaled by the entries of the diagonal matrix $\Sigma$, are obtained [28].

Since $U$ and $V$ are unitary, in contrast to linear (pre-)equalization neither transmit nor noise power are increased. However, since some singular values $\varsigma_n$ may be very small, independent signaling over the parallel channels using the same constellation and average power will show similar poor performance as linear equalization [31]. SVD-based transmission schemes should only by used in combination with optimized power (and rate) distribution of/and channel coding. Noteworthy, SVD is one of the essential steps in calculating the capacity of the MIMO channel.

2) Temporal Equalization: Temporal equalization based on eigendecomposition can be done if the transmission of blocks of $D$ symbols is considered. Subsequent blocks are separated.
by at least $D_g \geq L$ symbols (so-called guard interval), such that no ISI from the previous block reaches the current block. At the transmitter, a block $a$ of $D$ data symbols is transformed into transmit symbols by $x = VA$, which—after parallel/serial (P/S) conversion and insertion of the guard space (“+ guard”)—are transmitted in sequence. At the receiver the guard space may be ignored (“- guard”) and after serial/parallel (S/P) conversion, the decision vector is obtained as $r = U^H y$. The respective communication system is depicted in Figure 4.

Two basis schemes can be distinguished: during the guard period either zeros or the $D_g$ last samples of each block can be transmitted. Applying zeros, a scheme sometimes denoted as vector coding or structured channel signaling [25], [22], [24] is obtained, whereas in case of a cyclic prefix, orthogonal frequency division multiplexing (OFDM) or discrete multitone (DMT) [3] results.

Given the channel transfer function $H(z)$, first an equivalent channel matrix $H$ is set up. In case of zero guard space, the matrix has dimensions $(D + D_g) \times D$ and Toeplitz structure. The $D \times D$ transmit matrix is obtained by solving the following eigenvector problem [22], [24]

$$H^H H V = V \Sigma,$$  

where $\Sigma$ is a diagonal matrix. It can be shown, since $H^H H$ is a hermitian matrix, $\Sigma$ is real and $V$ is unitary. The $D \times (D + D_g)$ receive matrix (here the entire receive block is utilized) calculates to $U^H = V^H H^H$. In total, $D$ parallel, independent channels, scaled by the entries of $\Sigma$ are obtained.

In case of cyclic prefix and ignoring the respective portion of the receive sequence, the equivalent channel $H$ has dimension $D \times D$ and is a cyclic matrix; the rows (columns) are cyclic shifts of each other. Solving the eigenvector problem (10) results in $V$ to be the (normalized) inverse discrete Fourier transform (DFT) matrix (which is unitary), i.e., $V = W_D^H$, with $W_D \equiv 1 / \sqrt{D} \cdot [e^{-j2\pi ij/D}], i, j = 0, \ldots, D - 1$, independent of the actual channel. The receive matrix is the DFT matrix: $U^H = W_D$. Again, $D$ parallel, independent channels, scaled by the entries of $\Sigma$ are present. Due to signal generation via inverse DFT matrix and receive processing after transformation via DFT, the parallel channels may be interpreted as modulated on different carriers. Hence, this scheme is denoted as multicarrier modulation or OFDM/DMT.

The main advantage of OFDM/DMT over vector coding is that transmit and receive matrices are independent of the actual channel. Moreover, restricting $D$ to be a power of 2, very efficient implementations by fast transforms are possible. Since transmit power (and bandwidth) has to be spent for the cyclic prefix, and part of the receive energy is disregarded, OFDM/DMT exhibits a (slight) loss compared to optimal block transmission over the ISI channel.

3) Spatial/Temporal Equalization: The idea of using SVD or eigenvalue decomposition, respectively, for transmission over interference channels can be extended to joint spatial/temporal equalization.

The immediate approach would be to consider blocks of transmit and receive vectors and to build (large) vectors containing the samples of all antennas at all time instances during one block. The effective channel matrix has then block Toeplitz structure and is composed of the tap matrices $H_k$ of (2). Unfortunately, huge vectors and matrices have to be handled. Moreover, solving (10) again results in channel dependent transmit and receive matrices which can not be realized efficiently.

A more suited procedure is to do spatial/temporal equalization in two steps. First, temporal equalization is performed by applying OFDM transmitters and receivers, as shown in Figure 4, to each transmit and receive antenna, respectively. As a result, the MIMO ISI channel is decomposed into a set of parallel, independent flat fading MIMO channels, which are obtained from (2) as

$$H^{(d)} = H(e^{j2\pi(d-1)/D}), \quad d = 1, \ldots, D.$$

Second, to each of the parallel channels, spatial equalization is applied individually. In principle, this can be done by any of the discussed technique. However, of particular interest are decision-feedback equalization (BLAST) or SVD.

A straightforward application of the two steps will usually result in unsatisfactory performance. Some of the carriers/parallel channels will exhibit high error rates, which then dominates average error rate. Using an optimized rate and power distribution over space (antennas) and frequency (carriers), the same performance as for joint spatial/temporal DFE can be expected [13].
D. Lattice-Basis-Reduction-Aided Equalization

The equalization methods presented up to now are based on well-known principles and concepts from signal processing. Further improvements in performance are possible using “mathematical tools” not yet standard in communications. We will now discuss recent developments and possible future directions.

1) Spatial Equalization: Recently, a new approach for communication over flat-fading MIMO channels has been presented in literature [35], [31], [32]. The idea is——assuming signal sets taken from the Cartesian grid (not PSK constellations)—to describe the set of (noise-less) signal points at the channel output by the mathematical structure of a lattice. The channel matrix $\mathbf{H}$ is regarded as the generator of basis matrix of the lattice, i.e., the column vectors of $\mathbf{H}$ span the lattice. Form lattice theory it is known that each lattice can be described by different bases. Of particular importance is a so-called reduced basis, where the basis vectors have (i) norms as small as possible and (ii) are as close as possible to pairwise orthogonality. Algorithms for the generation of such a reduced basis are known, e.g., the Lenstra-Lenstra-Lovász (LLL) and the Korkine-Zolotareff (KZ) algorithms [23], [5]. Given such a reduced basis, traditional low-complexity detectors—e.g., simple linear equalization—are employed. Astonishingly, the error rate curves using such approaches run parallel to those for maximum-likelihood (ML) detection [31].

In order to apply lattice basis reduction algorithms to the current setting, the complex-valued MIMO channel model $\mathbf{H}$ is equivalently written as a real-valued one $\mathbf{H}_{\text{red}}$ by stacking real and imaginary parts of the vectors. Both models are equivalent and can be used interchangeably. Then, lattice basis reduction is applied which factors $\mathbf{H}$ according to [23], [5]

$$\mathbf{H} = \mathbf{H}_{\text{red}} \mathbf{R},$$  \hspace{1cm} (12)

where $\mathbf{R}$ is a matrix with integer entries that has unit determinant, i.e., $\mathbf{R}^{-1}$ also contains only integer entries. The matrix $\mathbf{H}_{\text{red}}$ is the reduced basis; $\mathbf{H}$ and $\mathbf{H}_{\text{red}}$ described the same lattice.

In digital communications the signal points in each quadrature component are usually drawn from (a translate of) the integer lattice $\mathbb{Z}$. Hence, at the receiver side the lattice $\mathbf{H}_{\text{red}} \mathbb{Z}^{2K} \equiv \mathbf{H}_{\text{red}} \mathbb{Z}^{2K}$ is present. Instead of linear equalization of $\mathbf{H}$, only the part $\mathbf{H}_{\text{red}}$ is linearly equalized. Since $\mathbf{R} \mathbb{Z}^{2K} = \mathbb{Z}^{2K}$, the (noise-less) decision symbols are drawn from the integer grid, and individual threshold decision of each component to the integer grid can be performed. Because the noise enhancement by $\mathbf{H}_{\text{red}}^{-1}$ is (typically much) lower than that of $\mathbf{H}^{-1}$, a gain in performance is achieved. In a last step, the data symbols are recovered via $\mathbf{R}^{-1}$. The transmission scheme using lattice-basis-reduction-aided equalization is depicted in Figure 5.

This principle of suited partial equalization, decision, and final residual equalization can be extended in various ways. First, linear equalization via $\mathbf{H}_{\text{red}}^{-1}$ can be replaced by DFE which offers some additional gain. Second, an MMSE version of lattice-basis-reduction-aided equalization has been proposed in [34]. Moreover, receiver side equalization can be transformed to transmitter side pre-equalization, which results in low-complexity schemes with astonishingly good performance. For details see [31].

2) Temporal and Spatial/Temporal Equalization: It is consequent to apply the idea of lattice-basis-reduction-aided equalization to temporal and spatial/temporal equalization. This can, in a first approach, be done by considering block transmission over the ISI channel (see last section) and applying lattice basis reduction the resultant Toeplitz or cyclic matrices.

However, for (SISO) ISI channels it is more desirable to directly factor the channel transfer function in the following way, analogous to (12)

$$H(z) = H_{\text{red}}(z) R(z),$$ \hspace{1cm} (13)

where $H_{\text{red}}(z)$ is some “reduced” transfer function and $R(z)$ has to have integer coefficients.

First results of ongoing research activities on this topic are very encouraging. Using a suited factorization (13) in combination with linear partial equalization of $H_{\text{red}}(z)$ shows very good performance. The main advantage of this approach is that—in contrast to DFE—channel coding can be applied much more easily.

E. Summary

In Table I the discussed equalization strategies for space and time are collected. For each strategy, the required transmitter and receiver side processing is given. Moreover, the mathematical principle or tool, on which the respective equalization strategy is based, i.e., what problem has to be solved when calculating the required matrices/filters, is given, too. The required mathematical tool ranges from simple matrix inversion over (sorted) QR-type decomposition or Cholesky factorization and spectral factorization of scalar and matrix polynomials to the theory of calculating a reduced basis for a lattice.

IV. Simulation Results

In this section, the performance of various combined spatial/temporal equalization approaches are compared by numerical simulations. In each case a $4 \times 4$ ($N_T = N_R = 4$) block-fading MIMO ISI channel is assumed. The power-delay profile of the $T$-spaced model is constant over $L = 4$ taps and the elements of the tap matrices $\mathbf{H}_k$ are chosen i.i.d. complex Gaussian with variance 1/4 (same average energy as a flat-fading MIMO channel). The numerical results—bit error rate (BER) over the ratio of total average transmitted energy per information bit $E_b$ and one-sided noise power spectral density $N_0$ of the channel—are averaged over a large number of channel realizations.

In Figure 6 the performance of (i) spatial/temporal DFE, (ii) OFDM (temporal) with DFE in each carrier (spatial), and (iii) OFDM (temporal) with lattice-basis-reduction-aided (linear) equalization in each carrier (spatial) are compared. In all cases, independent, uncoded 16-QAM signaling in each of the parallel channels is used. OFDM uses $D = 256$ carriers and loss due the guard interval is neglected. The
dashed curves show the respective genie-aided performance, i.e., perfect feedback in the DFE loop is assumed.

Unfortunately, using OFDM the temporal diversity offered by the channel (due to the ISI within each transmission burst) cannot be utilized—the error curve equals that of a flat-fading MIMO channel. This holds for DFE (BLAST) in each carrier (diversity order 1), as well as for lattice-basis-reduction-aided (LRA) equalization where the curve shows a diversity order of $N_R = 4$. Using spatial/temporal DFE temporal diversity is exploited and the slope of the error rate curve is much larger (in theory $N_R \cdot L = 16$). Via the feedforward filter, the receive signals of all antennas and the post cursors of the ISI channel (channel echos, temporal dimension) are combined constructively, which leads to the increased diversity order compared to OFDM. However, the error propagation in the feedback loop is much more severe as errors here propagate over space and time, whereas in OFDM each carrier is treated separately.

**TABLE I**

OVERVIEW OVER STRATEGIES FOR SPATIAL, TEMPORAL AND JOINT SPATIAL/TEMPORAL EQUALIZATION. CHARACTERIZATION OF REQUIRED TRANSMITTER (TX) AND RECEIVER (RX) SIDE PROCESSING, RESPECTIVELY. THE REQUIRED MATHEMATICAL TOOLS ARE GIVEN, TOO.

<table>
<thead>
<tr>
<th>Tool</th>
<th>TX: —</th>
<th>RX: $H(z)$</th>
<th>TX: —</th>
<th>RX: $H(z)$</th>
<th>TX: —</th>
<th>RX: $H(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear equalization at the receiver</td>
<td>TX: —</td>
<td>RX: $H(z)$</td>
<td>TX: —</td>
<td>RX: $H(z)$</td>
<td>TX: —</td>
<td>RX: $H(z)$</td>
</tr>
<tr>
<td>Linear pre-equalization at the transmitter</td>
<td>TX: $\xi H(z)$</td>
<td>RX: scalar scaling $\varsigma^{-1}$</td>
<td>TX: $\xi H(z)$</td>
<td>RX: scalar scaling $\varsigma^{-1}$</td>
<td>TX: $\xi H(z)$</td>
<td>RX: scalar scaling $\varsigma^{-1}$</td>
</tr>
<tr>
<td>Tool</td>
<td>matrix (pseudo, left/right) inverse</td>
<td>inverse transfer function</td>
<td>(pseudo, left/right) inverse matrix transfer function</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decision-feedback equalization at the receiver</td>
<td>TX: —</td>
<td>RX: feedforward matrix $F$, feedback matrix $B$, permutation matrix $P$</td>
<td>TX: —</td>
<td>RX: feedforward filter $F(z)$, feedback filter $B(z)$, permutation matrix $P$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Precoding at the transmitter</td>
<td>TX: permutation matrix $P$, feedback matrix $B$, feedforward matrix $F$</td>
<td>RX: scalar scaling $\varsigma^{-1}$</td>
<td>TX: feedforward filter $B(z)$, feedback filter $F(z)$</td>
<td>RX: scalar scaling $\varsigma^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tool</td>
<td>sorted QR-type decomposition or Cholesky factorization</td>
<td>spectral factorization</td>
<td>sorted factorization of matrix polynomials</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Singular value decomposition Vector coding, OFDM/DMT</td>
<td>TX: unitary matrix $V$</td>
<td>RX: unitary matrix $U^H$, scaling matrix $\Sigma^{-1}$</td>
<td>TX: inverse DFT matrix $W_D^H$</td>
<td>RX: DFT matrix $W_D$, scaling matrix $\Sigma^{-1}$</td>
<td>TX: processing per carrier, inverse DFT matrix $W_D^H$ for each antenna</td>
<td></td>
</tr>
<tr>
<td>Tool</td>
<td>SVD</td>
<td>eigenvector decomposition diagonalization of cyclic matrices</td>
<td>combination of SVD and diagonalization of cyclic matrices</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lattice-basis-reduction-aided equalization</td>
<td>TX: —</td>
<td>RX: partial equalization via $H_{red}$ quantization to integers residual equalization via $R^{-1}$</td>
<td>(topic of current research)</td>
<td>(topic of current research)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each antenna processing per carrier.
Fig. 6. Bit error rate over signal-to-noise $E_b/N_0$ (in dB) for OFDM (temp.) / DFE (spat.), spatial/temporal DFE, and OFDM (temp.) / LRA (spat.). $L = 4$ tap channel with constant power-delay profile, $N_T = N_R = 4$, 16-QAM modulation and equal power for each data stream.

Fig. 7. Bit error rate over signal-to-noise $E_b/N_0$ (in dB) for OFDM/DFE (temp./spat.) and spatial/temporal DFE. Without (16-QAM modulation) and with optimized rate and power distribution (same total transmission rate). $L = 4$ tap channel with constant power-delay profile, $N_T = N_R = 4$. 
When using optimized rate and power loading, the situation changes. The respective error rate curves using the loading algorithm in [10] are plotted in Figure 7. Here, the signal constellations are restricted to square QAM sets with sizes $2^z$, $z \in \mathbb{N}$. As can be seen, DFE improves only slightly by loading, whereas significant gains are achieved in OFDM/DFE. It can be shown analytically [13] that neglecting error propagation OFDM/DFE and joint spatial/temporal DFE perform the same (the respective error rate curves in Figure 7 are almost indistinguishable).

V. SUMMARY AND CONCLUSIONS

In this paper an overview on spatial, temporal, and joint spatial/temporal equalization strategies for point-to-point communication over flat-fading MIMO channels, ISI channels, and MIMO ISI channels, respectively, is given. Each approach (linear, decision-feedback, and eigenvector/SVD-based equalization) can successfully applied to either spatial, temporal, or joint processing. The similarities and differences between the schemes has been enlightened. Numerical results reveal that a careful design of OFDM/DFE or spatial/temporal DFE will lead to similar performance in space/time transmission.

Moreover, the mathematical principles of which the equalization schemes are implementations, are specified. Each schemes is a particular instant of a general mathematical problem. Applying “new” mathematical tools, not yet common in communications, may lead to new, hopefully low-complex and high-performance transmission systems. The recently introduced lattice-basis-reduction-aided equalization schemes are examples for utilizing successfully an abstract, in mathematics long-known principle, now in digital transmission over MIMO channels. The extension to ISI channels is a challenging task for future research.

REFERENCES


