CONSTRUCTION OF LOW–RATE POWER–EFFICIENT CODING SCHEMES AND THEIR APPLICATION TO CDMA

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Abstract — In this paper we propose a novel design method, which is an extension to the EXIT charts introduced in [5], for low–rate power–efficient codes. An exemplary design of a rate–1/4 code shows more than 1 dB more coding gain than the original turbo–code [2] of rate–1/2. Using this code in a CDMA scheme while reducing the spreading by a factor of 2 helps improving the overall performance of the multi–user scheme significantly.

I. Introduction
Since the invention of turbo–codes in 1993 [2] unveiling the iterative decoding algorithm used has been a vivid field of research. Recently a breakthrough has been independently obtained by several authors [5, 16, 10, 9] that all analyzed concatenated coding schemes solely by characterizing the behavior of the component codes. In the present paper, we extend the method of EXIT charts [5] to the analysis of low–rate parallel concatenated code. Exemplary a rate–1/4 code is shown to perform extremely close to the capacity limit. This code has the potential to more than double the number of users in a CDMA scheme, when compared to the original turbo–code [2].

II. Multiple–Turbo–Codes
The codes considered in this paper consist of parallel concatenations of convolutional codes. The constituent encoders, which in general may not be identical and may not have identical rates, are fed with a differently interleaved version of the input bits \( \vec{U} = \{U_1 \ldots U_K \} \). This structure has been introduced as turbo–code in [2], where the concept was limited to two parallel constituent encoders, and extended to the concatenation of three or more constituent codes in [8].

In the following we restrict ourselves to use constituent codes of code rate \( R = 1 \), only. We are especially interested in low complexity coding schemes. Hence, only to use scramblers \( \text{scr} \), as rate–1 codes are usually called in the context of turbo–codes, seems to be reasonable, as scramblers achieve the lowest rate for a given number of constituent codes. But, the extension of our method to constituent codes of other rates is straightforward.

For convergence of iterative decoding it is mandatory to have at least two recursive constituent codes. Hence, symmetric multiple–turbo–codes can only be build from recursive component codes. But, asymmetric structures can also incorporate feedforward–only convolutional codes [7].

Fig. 1 shows an encoder for a rate–1/4 multiple–turbo–code build from three scramblers.

In addition to the scramblers we see in Fig. 1 that there is one more important building block in the encoder. The interleavers generate two important properties of the turbo–codes. Firstly, they decorrelate the input sequences to the scramblers. This enables iterative decoding. Furthermore, the interleavers determine the minimum distance, which is the dominating property in the high signal–to–noise regime. But, as we only are inter-
ested in asymptotical analysis and design of codes with very large interleavers, that nearly reach the asymptotical limit of infinite interleaving, there is no impact of interleaver design. Hence, we use interleavers that perform random permutations. For all purposes, it is absolutely mandatory that the interleavers are different. The length of the interleavers, denoted by $K$, determines the number of information bits encoded into one turbo-code word.

As maximum-likelihood decoding of turbo-codes is in general computationally prohibitive [3], we investigate turbo-codes together with an iterative decoding algorithm performing (suboptimal) decoding with reasonable complexity. Instead of taking the code constraints of all constituent codes simultaneously into account, each constituent code is decoded separately and passes its output to the other constituent decoders. The message-passing strategy used for iterative decoding is depicted in Fig. 2. It is called extended serial decoding structure [12].

![Figure 2: Extended serial decoding structure.](image)

This structure gives the best performance and is computationally very efficient. Each decoder passes its extrinsic information to all other decoders and receives the extrinsic information from all other decoders. As soon as an updated extrinsic information is available, it is passed to all decoders. After a number of iterations (typically 8, 16 or 32), each consisting of decoding every constituent code once, the iterative process is stopped and the information sequence is estimated.

As component decoders BCJR-algorithms [1] are used. These decoders perform optimum symbol-by-symbol soft-input/soft-output decoding of convolutional codes and can take into account a-priori knowledge on the information bits.

III. Convergence Analysis of Asymmetric Multiple-Turbo-Codes

The convergence analysis as proposed here is based on modeling the components of the iterative decoder using transfer characteristics of the component decoders and an approximation of information combining. Our analysis method is an extension to the EXIT charts introduced in [5]. EXIT charts permit a detailed investigation of the asymptotic behavior of concatenated coding schemes consisting of two components (asymptotic with respect to interleaver size). In a simplified model for the iterative decoding process it is assumed that the a-priori and extrinsic information can be characterized by jointly independent, identical distributed random variables. Then, the decoder can be characterized by the transfer from given a-priori information $Z$ measured by the mutual information $I(U; Z)$ between source data $U$ and a-priori information $Z$, to extrinsic information $E$, also measured by its mutual information $I(U; E)$ to the source data $U$. Mutual information is always measured symbol-by-symbol, as independence is assumed due to sufficient interleaving. For large interleavers the extrinsic information is independent relative to decoding horizon over a large number of iterations, and hence, this model is in excellent agreement with measurements in concatenated coding schemes.

The transfer characteristics

$$I(U; E) = T[I(U; Z)]$$

of the individual component codes are determined without any knowledge about the other component codes. The a-priori information $Z$ can approximately be modeled as a Gaussian random variable, because the individual probability density function has almost no effect on mutual information, cf. [14]. Therefor, the decoder is fed with the symbols $Y$ received from the physical additive white Gaussian noise (AWGN) channel and the side information transmitted over an additional AWGN side channel. Mutual information between the source symbols and the a-priori input $I(U; Z)$ as well as between the source symbols and the extrinsic output $I(U; E)$ provided by the decoder is measured. The transfer characteristics $I(U; E) = T[I(U; Z)]$ are parameterized by the signal-to-noise ratio $E_b/N_0$ of the channel. Fig. 3 shows a block diagram of a measurement setup operating in the Log-domain [11].

![Figure 3: Measurement setup to determine the transfer characteristic of a scrambler scr.](image)
the a-priori information of the second decoder, whereas the extrinsic output of the second decoder becomes the a-priori information of the first decoder. This can be depicted in a graph using the exchanged informations as axes, which simultaneously shows the transfer characteristics of both constituent codes at the given signal-to-noise ratio $E_b/N_0$. The resulting EXIT charts provide a visualization of the iterative decoding process. A zigzag–path of horizontal and vertical lines that start at one of the transfer characteristics and end at the transfer characteristic of the other decoder show the evolution of extrinsic information during iterative decoding. If the upper right corner in the EXIT chart is reached by this zigzag–path, error-free communication is possible for the used component codes if there is no delay constraint. This is called convergence of iterative decoding. But, if the transfer characteristics intersect, the bit error rate of the analyzed coding scheme can at this particular signal-to-noise ratio be lowered to zero neither by increasing the interleaver depth nor by additional iterations.

Whereas in the decoder of an original turbo-code information between the two decoder is exchanged directly, in a decoder for a multiple–turbo–code additional signal processing is performed, that has to be modeled for the convergence analysis. E.g., in a rate–1/4 multiple–turbo–code, cf. Fig. 2, decoder 1 gets information from decoder 2, decoder 3 and decoder 4. Due to interleaving it can be assumed that the a-priori information for a special bit obtained from decoder 2 is independent of the output of decoder 3 and decoder 4. Hence, within the decoder maximum–ratio combining, which is the optimum data processing [4], can be performed via addition of the Log–Likelihood–Ratio–soft–output of the two decoder.

But, for the convergence analysis we have to model the decoder statistically. The a-priori information $I_{A1}$, which decoder 1 receives, is the result of a combination of both extrinsic informations $I_{E2}, I_{E3}$ and $I_{E4}$ from decoders 2 and 3, in general:

$$I_{A1} = f(I_{E2}, I_{E3}, I_{E4}) \quad (2)$$
$$I_{A2} = f(I_{E1}, I_{E3}, I_{E4}) \quad (3)$$
$$I_{A3} = f(I_{E1}, I_{E2}, I_{E4}) \quad (4)$$
$$I_{A4} = f(I_{E1}, I_{E2}, I_{E3}) \quad (5)$$

In [14] it was shown that the dependence of $I_{A1}$ on $I_{E2}, I_{E3}$ and $I_{E4}$ can be closely approximated. This can be done by modeling the decoder outputs as noisy transmissions of $U$ over parallel independent channels. In the following we study the information combining of two sources and then recursively determine the function $f(\cdot, \cdot, \cdot)$, e.g.,

$$f(I_{E2}, I_{E3}, I_{E4}) = g(I_{E2}, g(I_{E3}, I_{E4})) \quad (6)$$

To model $g(\cdot, \cdot)$, e.g., we may apply two parallel binary symmetric channels (BSCs) with identical inputs having bit error ratios $\epsilon_1$ and $\epsilon_2$, see Fig. 4.

Figure 4: Model of two parallel BSC.

Equivalently to two parallel BSCs a memoryless discrete channel with binary input and quaternary output, cf. Fig. 5, can be used.

Figure 5: Discrete memoryless channel equivalent to two parallel BSC.

Its capacity is given by:

$$C_{sum} = I(U; E_1 E_2) = I(U; Z)$$
$$= 1 - e_2(\epsilon_1) - e_2(\epsilon_2) - e_2((1-\epsilon_1)(1-\epsilon_2)+\epsilon_1\epsilon_2). \quad (7)$$

Here, $e_2(\cdot)$ denotes the binary entropy function $e_2(x) := -x \log_2(x) - (1-x) \log_2(1-x)$, $x \in (0, 1)$.

For a further example, the information may be modeled as transmitted over two parallel AWGN channels with a single binary input using binary shift keying (BPSK) for signaling, cf. Fig. 6.

Figure 6: Model of two parallel BPSK–AWGN channels.
As maximum–ratio combining is optimum [4], the two parallel BPSK–AWGN channels result in a single equivalent BPSK–AWGN channel with noise variance $\sigma^2 = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$. Hence, the equivalent channel capacity is

$$C_{\text{sum}} = I(U; E_1E_2) = I(U; Z)$$

$$= C_{\text{BPSK}} \left(\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right).$$\hspace{1cm}(8)$$

The computationally most efficient way to model the function $g(\cdot, \cdot)$ is to use two parallel binary erasure channels (BECs). As their capacity linearly depends on the erasure rate $p$ ($C = 1 - p$) and the resulting equivalent channel also is a BEC, the “sum–capacity” can be determined just by multiplication and addition:

$$C_{\text{sum}} = I(U; E_1E_2) = I(U; Z)$$

$$= 1 - (1 - I(U; E_1))(1 - I(U; E_2)).$$\hspace{1cm}(9)$$

As the results of (7), (8) and (9) are very similar, we can state, that for the information theoretic treatment the specific channel model does not play a key role. If we only know the capacity of the parallel channels and have no knowledge on the detailed behavior, we still can calculate the “sum–capacity” with sufficient accuracy.

With the transfer characteristics and the knowledge how to model the information combining we are able to analyze iterative decoding of multiple–turbo–codes.

Our algorithm for the analysis of the convergence behavior of a particular multiple–turbo–code works as follows:

1) We set $I_{A1} = \epsilon$ to a small positive number and all extrinsics $I_{Ej}$ to zero. By this starting condition we can ensure, that even if all constituent codes which are rate–1 scramblers, are catastrophic, i.e. have a rational encoder inverse, convergence is possible in principle. If all transfer characteristics start at the origin $T[I_A = 0] = 0$, systematic doping [6] has to be used to trigger iterative decoding.

2) We evaluate the transfer characteristic $T$ of a component code at $I_A$: $I_E = T(I_A)$.

3) We calculate the input to the next constituent decoder by determining the “sum–capacity” from the extrinsic information output of all other decoders, e.g., for three components $I_A = g(I_{E1}, I_{E2})$ holds.

4) We repeat steps 2) and 3) for all constituent codes.

5) We repeat steps 2) to 4), that form a full iteration, until the a–priori information for all decoders remains constant.

If any of the a–priori information converges to 1.00, convergence of iterative decoding is possible, i.e. with infinitely long interleavers reliable communication is possible for the analyzed signal–to–noise ratio $E_b/N_0$.

IV. DESIGN EXAMPLE FOR RATE–1/4

Using the algorithm given above for the convergence analysis of multiple–coding–schemes, we are able to analyze 8855 (asymmetric) multiple–turbo–codes using the 20 transfer characteristics of potential constituent codes up to memory $m = 3$. There are 15 different recursive scramblers and 5 feedforward–only scramblers, whose transfer characteristics have to be determined. Hence, the code search over all 8855 codes is very fast.

The results of this convergence analysis are given in Table 1. Additionally to the generator polynomials for all three scramblers, the minimum number of iterations needed to achieve a bit error ratio smaller than $10^{-7}$ (assuming infinite interleaving) is given. Multiple–turbo–codes that require systematic doping [6] are marked (*).

| Table 1: Some rate–1/4 multiple–turbo–codes with convergence at $E_b/N_0 < -0.6$ dB |
| --- | --- | --- | --- | --- |
| SCR 1 | SCR 2 | SCR 3 | SCR 4 | min Iter |
| 1* | 03 | 03 | 05/07 | 013/015 | 20 |
| 2* | 03 | 07 | 07/06 | 013/015 | 21 |
| 3* | 03/07 | 03/07 | 011 | 017 | 31 |
| 4 | 03 | 07 | 01/013 | 01/013 | 13 |

Figure 7: Performance of the rate $R = 1/4$ asymmetric multiple–turbo–code Nr. 1 (Block length $K = 100000$). For comparison the performance of the original turbo–code of rate–1/2 for the same block length is also given.
V. CDMA SCHEME IMPROVEMENT VIA RATE LOWERING

In order to demonstrate the practical relevance of low-rate power-efficient coding, we consider an idealized code division multiple access (CDMA) scheme. Within an uplink–scenario asynchronous users communicate simultaneously with the same base–station over an AWGN channel sharing the same frequency band. At the receiver side linear minimum mean square error (MMSE) detection is assumed. We only consider a “single cell” system, i.e., there is no interference from other sources than the users and the additive noise. As, for short spreading sequences, it is possible for the signals of two asynchronous users to be highly correlated, we only allow long (random) spreading sequences. The fixed rate $L$ between the users is required, to allow doubling the load.

The spreading sequences, it is possible for the signals of two asynchronous users to be highly correlated, we only allow long (random) spreading sequences. The fixed rate $L$ between the users is required, to allow doubling the load.

As a result we can see, that, e.g., at a channel $SNR = -2$ dB the load can be increases from $\zeta = 0.16$, which is the maximum load possible if $SNR_{MMSE} > 0.6$ dB is required, to $\zeta = 0.63$, if $SNR_{MMSE} > -0.5$ dB is sufficient. Including the rate–loss of the low–rate coding scheme we still get more than a doubled number of users.

VI. CONCLUSION

In this paper we have shown a novel method for the design of low-rate power-efficient codes. Using well known techniques, as used to construct EXIT charts, and a new model for information combining we were able to analyze asymmetric multiple–turbo–codes of any rate. In an exemplary construction of a rate–$1/4$ code it has been shown, that with this design method the capacity limit can be approached very closely.

Low-rate power-efficient codes are shown to be very practical if CDMA schemes are used. For low signal–to–noise ratio coding has a large advantage over spreading, with additional coding gain is realized. An example for linear MMSE detection shows significant improvement in the number of users if the new low-rate coding scheme is compared to the original turbo-code.

REFERENCES


