Directed Selected Mapping for Peak-to-Average Power Ratio Reduction in Single-Antenna OFDM

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Abstract—The high peak-to-average power ratio of the transmit signal is a serious drawback of multicarrier transmission. To cope with this problem, peak power reduction schemes have been developed. One popular method is selected mapping (SLM), where several alternative representations are generated from the information carrying signal and that one which exhibits the lowest peak power is selected for transmission. Increasing the number of alternative candidates, i.e., spending higher computational complexity, better peak power reduction can be achieved. In this paper, a novel approach of SLM is presented, which considers a number of consecutive OFDM frames, a hyper frame, jointly. A trade-off between complexity and introduced delay is possible. Choosing an appropriate combination of number of assessed candidates and hyper frame lengths significant gains over conventional SLM can be achieved.

I. INTRODUCTION AND SYSTEM MODEL

In wireless communication systems the application of multicarrier modulation is common for digital transmission over frequency-selective channels. In particular, orthogonal frequency-division multiplexing (OFDM) is a very popular implementation.

One serious drawback of OFDM is the high peak-to-average power ratio (PAR) of its transmit signal. Due to non-linear amplifiers in the transmitter a high PAR leads to signal clipping which causes transmission errors and, even worse, out-of-band radiation. To overcome these issues it is necessary to control the transmit signal in order to reduce its PAR.

For many years, peak-power reduction algorithms for single-antenna systems (also denoted as single-input/single-output (SISO) systems) have been an important field of research. Various methods have been developed, e.g., partial transmit sequences (PTS) [1], active constellation extension [2], or tone reservation [3], [4]; for an overview see [5]. One of the most popular techniques for PAR reduction is selected mapping (SLM) [6]. SLM and PTS are both based on multiple signal representation.

Recently, the application of OFDM together with multiple transmit antennas (often denoted as MIMO OFDM) has been discussed. In this context an extension of SLM, namely directed SLM (dSLM) was proposed [7], [8] which is able to utilize the additional degree of freedom offered by multiple transmit antennas. Thereby, significant gains in PAR reduction compared to a straightforward application of SLM to MIMO systems can be achieved. In this paper we apply the ideas of [7], [8] to single-antenna systems.

Throughout this paper, we consider a discrete-time OFDM system model. The binary data stream is mapped (mapping $\mathcal{M}$) to (complex-valued) data symbols taken from an arbitrary $M$-QAM or $M$-PSK constellation. A frequency-domain OFDM frame is constructed of $D$ data symbols and denoted by the vector $A = [A_d]$. This frequency-domain OFDM frame $A$ is transformed into a time-domain OFDM frame $a$ via an inverse discrete Fourier transform (IDFT) of length $D$, defined by $a_k = \frac{1}{\sqrt{D}} \sum_{d=0}^{D-1} A_d \cdot e^{j2\pi kd/D}$, $k = 0, \ldots, D - 1$. Assuming statistical independent frequency-domain symbols $A_d$, due to their superposition within the IDFT, the elements of the time-domain vector are almost Gaussian distributed.

As performance measure for PAR reduction we resort to the complementary cumulative distribution function $\operatorname{ccdf} \equiv \Pr\{\text{PAR} > \text{PAR}_{\text{th}}\}$, which gives the probability that the PAR of the current OFDM frame exceeds a certain threshold $\text{PAR}_{\text{th}}$. Thereby we consider the PAR of the discrete-time signal, i.e.,

$$\operatorname{PAR} \equiv \frac{\max_{k=0,\ldots,D-1} |a_k|^2}{E\{|a_k|^2\}}. \quad (1)$$

This measure is common in literature and sufficient for a fair comparison of different PAR reduction techniques. We also do not consider the cyclic prefix, pulse shaping, or modulation to radio frequency.

Assuming the time-domain samples $a_k$ to be Gaussian distributed, the ccdf of the original OFDM scheme is given by [6]

$$\operatorname{ccdf}_{\text{orig}} \equiv 1 - \left(1 - e^{-\text{PAR}_{\text{th}}}ight)^D. \quad (2)$$

In Section II, a brief review of different SLM approaches is given. Section III introduces the new dSLM approach for single-antenna OFDM and in Section IV analytic expressions for the ccdf are derived. In Section V, performance results of the different SLM approaches are compared; Section VI draws some conclusions.

II. REVIEW OF STATE-OF-THE-ART SLM

A. Original SLM Approach

The idea of SLM is to generate multiple, say $U$, different representations from the information carrying frequency-domain OFDM frame $A$. These $U$ signal representations may be obtained by an element-wise multiplication of $A$ by $U$ different, randomly selected phase vectors $P$. According to [6] the elements of $P$ may be chosen from the set $\{\pm1, \pm j\}$, which gives good results in PAR reduction and does not affect the receiver sided synchronisation.

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Next, the \( U \) different signal representations are transformed into time domain and that one exhibiting the lowest PAR is selected for transmission. Thus, a total computational complexity of \( U \) IDFTs and \( U \) metric calculations (1) is necessary. In order to decode the signal correctly, the receiver must be informed which one of the \( U \) possible signal representations was transmitted. Assuming both, transmitter and receiver aware of a codebook of possible phase vectors it is sufficient to transmit the index of the actual used phase vector. The side information is represented by \( \lceil \log_2(U) \rceil \) bits. In [9] a variant of SLM was proposed, which does not require the transmission of explicit side information.

In the top most subfigure of Fig. 1 a delay profile of SLM is depicted. For any computational calculations, e.g., calculation of IDFT or PAR, an infinite processing speed is assumed. The procedure of calculating the best time-domain samples from the original amplitude coefficients is done as follows. First, the frequency-domain OFDM frames (white) are grouped by using \( D \) length coefficients into one OFDM frame. Next, the OFDM frame is passed to the SLM algorithm. For each OFDM frame \( U \) alternative signal representations are assessed (illustrated by thin red lines). After the SLM algorithm the best time-domain OFDM frame (gray) is available and can be transmitted symbol-by-symbol. Thus, an inherent delay of one OFDM frame is present.

Assuming again Gaussian distributed time-domain samples, the ccdf of SLM is given by [6]

\[
\text{ccdf}_{\text{SLM}} \equiv (1 - (1 - e^{-\text{PAR}_{\text{aw}}})^{\frac{1}{D}U}.
\]

**B. Ordinary and Directed SLM for Multi-Antenna OFDM**

For future communication systems the use of multiple transmit antennas is envisaged in order to increase capacity [10]. The combination of OFDM with so called multiple-input/multiple-output (MIMO) OFDM.

An application of SLM to MIMO OFDM, called ordinary SLM, was first proposed in [11]. It is just a simple, independent implementation of the original SLM technique to each antenna. Performance is now governed by the worst-case PAR, i.e., the worst PAR over all transmit antennas. Due to this fact, PAR reduction gets a more complex task in MIMO communication and the PAR performance gets worse with an increasing number of transmit antennas.

In [7], a novel SLM technique, called directed SLM (dSLM), was proposed for MIMO OFDM. This technique considers the OFDM frames over all transmit antennas jointly. After calculating an initial PAR of all transmit antennas this method only treats that transmit antenna exhibiting the worst PAR. A pseudocode description of this algorithm is given in [7, Fig. 1]. Demanding fixed computational complexity, for each OFDM frame only on average \( U \) candidates are tested; more computational effort is spent for antennas with higher PAR, less for that with already lower PAR.

Directed SLM is able to utilize the multiple transmit antennas. PAR reduction performance gets better for an increasing number of antennas. This effect is similar to the diversity order when transmitting over flat-fading MIMO channels. In [8] an approximation of the ccdf was given; in Section IV we will derive an exact expression.

**III. DIRECTED SLM FOR SINGLE-ANTENNA OFDM**

We now apply the concept of MIMO dSLM to single-antenna systems, and refer to this method as SISO dSLM. The idea of the MIMO dSLM algorithm from [7, Fig. 1] is to consider several OFDM frames (over the spatial dimension) jointly. In single-antenna systems it is possible to group temporally consecutive OFDM frames. A block of \( N_t \) successive considered OFDM frames will be denoted as hyper frame.

**A. Non-Overlapping Hyper Frames**

The straightforward approach is to regard non-overlapping hyper frames, i.e., after each run of dSLM the window is shifted by \( N_t \) OFDM frames. As in SISO transmission all OFDM frames of one hyper frame are
transmitted in a consecutive order, not only the worst-case PAR is decisive here, but the PAR of all OFDM frames. Nevertheless, it is still desirable to reduce the PAR of the worst OFDM frame as this one dominates the result.

At most \( N_t(U - 1) + 1 \) different signal representations may be accounted for one OFDM frame. Hence, side-information of \( \lceil \log_2(N_t(U - 1) + 1) \rceil \) bits have to be transmitted.

The main drawback of the concept of hyper frames is that an extra delay \( \Delta \) of the OFDM frames compared to the original SLM approach is introduced. The corresponding delay profile is depicted in Fig. 1 (second subfigure). The SISO dSLM algorithm has to wait for \( N_t \) frequency-domain OFDM frames (white) to form a hyper frame. After evaluating \( U N_t \) (illustrated by thick red lines) alternative signal representations, the whole hyper frame is available in time domain (gray) and can be transmitted symbol-by-symbol. Compared to the original SLM approach an extra delay of \( \Delta = N_t - 1 \) OFDM frames arises.

As can be seen from Fig. 1, the computational capacity is used in an unsteady way, i.e., after each \( N_t \) th OFDM frame a large number of calculations has to be done; in the remaining time no calculations are necessary.

B. Overlapping Hyper Frames

In order to achieve a steady use of the computational capacity the observation window can also be shifted by one OFDM frame (sliding window approach). For the new OFDM frame one IDFT and PAR have to be calculated in order to obtain the initial value. Now, the remaining budget of \( U - 1 \) IDFTs and PAR calculations can be distributed on the whole hyper frame. In Fig. 1 (third subfigure) the delay profile is shown. After each new OFDM frame \( U \) alternative signal representations are assessed (illustrated by thin red lines) and the OFDM frame is available in both, frequency- and time-domain (white and gray) as both versions are necessary in the next step. The (best version of the) eldest OFDM frame is released for transmission. The extra delay is again \( \Delta = N_t - 1 \).

C. Overlapping with Reordering of the OFDM Frames

An extension to the overlapping SISO dSLM approach is to reorder the OFDM frames temporally. In particular, not the eldest OFDM frame may be released for transmission but the currently best one within the processing window. In order to avoid an infinite delay of OFDM frames we introduce a maximal additional delay \( \Delta_{\text{add}} \) after which each OFDM frame must be transmitted. The delay profile for a certain example and a maximal additional delay of \( \Delta_{\text{add}} = 2 \) is given by the bottom subfigure of Fig. 1. Compared to pure overlapping SISO dSLM additional memory (either at transmitter or receiver) is required in order to undo the reordering. Now, the extra delay sums up to \( \Delta = N_t - 1 + \Delta_{\text{add}} \).

IV. DERIVATION OF THE COMPLEMENTARY CUMULATIVE DISTRIBUTION FUNCTION

Exact formulas of the ccdf for SLM were already given in [6]. For the new SLM approach, presented in this paper, we now derive an analytic expression for the ccdf.

A. CCDF of the Original Signal

First, we derive the ccdf for the best, down to the worst PAR within one hyper frame. For convenience, \( \rho = \text{ccdf}_\text{orig} \) and \( X \equiv \text{PAR} \) are defined. The PARs of the OFDM frames within one hyper frame of length \( N_t \) are assumed to be sorted

\[
X_0 < \ldots < X_\lambda < \ldots < X_{N_t - 1},
\]

i.e., \( \lambda = 0 \) corresponds to the best and \( \lambda = N_t - 1 \) to the worst PAR. The ccdf of the PAR of the \( \lambda^{\text{th}} \) OFDM frame is given by

\[
\text{ccdf}_{\text{orig}, \lambda} = \Pr\{X_\lambda > X_0\} = \sum_{n=0}^{\lambda} \Pr\{X_{n-1} < X_\lambda < X_n\}, \quad X_0 < \ldots < X_{N_t - 1}
\]

The probability that the threshold \( X_\lambda \) lies between the sorted PARs for \( \lambda = n - 1 \) and \( \lambda = n \) is given by a binomial distribution with a single event probability of \( 1 - \rho \) and \( n \) events out of \( N_t \) possible ones come true:

\[
\Pr\{X_{n-1} < X_\lambda < X_n\} = \binom{N_t}{n}(1 - \rho)^n \rho^{N_t - n}, \quad n = 0, \ldots, N_t
\]

In summary, the probability (4) is given by

\[
\text{ccdf}_{\text{orig}, \lambda} = \rho^{N_t} \sum_{n=0}^{\lambda} \binom{N_t}{n}(1 - \rho)^n, \quad \lambda = 0, \ldots, N_t - 1
\]

The ccdf of the entire original OFDM signal is then given by the arithmetic mean over (6)

\[
\text{ccdf}_{\text{orig}} = \frac{1}{N_t} \sum_{\lambda=0}^{N_t-1} \text{ccdf}_{\text{orig}, \lambda}
\]

B. CCDF of Directed Selected Mapping

In case of non-overlapping dSLM (dSLM-no) not only \( N_t \) OFDM frames are considered but \( U \cdot N_t \) ones. Hence, the individual ccdfs are given by

\[
\text{ccdf}_{\text{dSLM-no}, \lambda} = \rho^{UN_t} \sum_{n=0}^{\lambda} \binom{UN_t}{n}(1 - \rho)^n
\]

Out of these alternative signal representations the \( N_t \) best ones are selected for transmission, thus the average ccdf is given by

\[
\text{ccdf}_{\text{dSLM-no}} = \rho^{UN_t} \sum_{\lambda=0}^{N_t-1} \binom{UN_t}{\lambda}(1 - \rho)^\lambda (N_t - \lambda)
\]
C. Asymptotic Behaviour of the CCDF

Using the approximation \((1-e^{-x})^y \approx 1 - ye^{-x}\), for \(x \gg 1\), leads to \(\rho \approx De^{-\text{PAR}_{\text{th}}\xi}\). For large values of \(\text{PAR}_{\text{th}}\), \(1 - \rho\) tends to one and the sum in (9) is dominated by its last term, i.e., \(\lambda = N_t - 1\). Hence, the ccdf can be approximated by

\[
\text{ccdf}_{\text{SLM-no}} \approx \left(\frac{U N_t}{N_t - 1}\right)^{\rho^{N_t(U-1)+1}/N_t}.
\]

(10)

In a semi-logarithmic scale, the ccdf (9) can hence be approximated by the linear function

\[\log(\text{ccdf}_{\text{SLM-no}}) \approx \text{const} - \text{PAR}_{\text{th}} \cdot (N_t(U - 1) + 1).\]

Thus, the asymptotic slope is given by \(N_t(U - 1) + 1\).

D. Excursus to Multi-Antenna OFDM

Using the formulas derived above, it is possible to give an exact expression for the ccdf of MIMO dSLM. Subsequently, we consider a multi-antenna system with \(N_T\) transmit antennas, which replaces \(N_s\). As for MIMO OFDM the worst-case PAR is decisive, only the ccdf of the worst PAR (i.e., \(\lambda = N_T - 1\)) of non-overlapping SISO dSLM (8) has to be regarded. Hence, the ccdf of MIMO dSLM is given by \(\text{ccdf}_{\text{MIMO dSLM}} = \text{ccdf}_{\text{SLM-no},N_t-1}\).

Analyzing the asymptotic behaviour of \(\text{ccdf}_{\text{MIMO dSLM}}\), as done above, the slope parameter is again \(N_T(U - 1) + 1\). This was already envisaged in [8] and approximated by \(UN_T\).

V. PERFORMANCE RESULTS

The subsequent performance results are based on a single-antenna OFDM transmission with \(D = 512\) carriers. The modulation alphabet is 4-QAM; however, the results are independent of the modulation alphabet and valid for any \(M\)-QAM or PSK constellation. Using the principle of widely-linear signal processing [12] the complexity can be further reduced to \(\sqrt{U}\).

Fig. 2 shows the ccdf of dSLM (non-overlapping and overlapping, no additional delay) for different values of alternative signal representations \(U\) and hyper frame lengths \(N_t\). Both, results of numerical simulations (colored) and the analytical results (gray) are depicted. The small differences emerge from the fact, that the time-domain OFDM signal is not perfectly Gaussian. Directed SLM provides a significant gain compared to original SLM (about \(1\text{ dB}\) for \(U = 8\) and \(N_t = 8\) at a clipping probability of \(\text{Pr}(\text{PAR} > \text{PAR}_{\text{th}}) \approx \text{Pr}_c = 10^{-5}\).

The performance of dSLM increases if more alternative signal representations are taken into account (as it does for the original SLM approach). This, however, leads to more computational complexity as the number of IDFT and PAR calculations increases as well. Considering a larger hyper frame length \(N_t\) improves PAR reduction performance as well, without increasing complexity. Now, a delay of \(\Delta = N_t - 1\) OFDM frames occurs.

The results of overlapping dSLM (dSLM-o) are slightly better than non-overlapping dSLM. In addition to that, it uses the computational capacity more steadily (see Fig. 1). According to (10), a larger hyper frame length \(N_t\) leads to a steeper decay of the ccdf for high values of \(\text{PAR}_{\text{th}}\). This can also be observed in Fig. 2.

In order to assess which parameter combination \((U, N_t)\) is preferably used in PAR reduction, Fig. 3 shows contour plots of the thresholds \(\text{PAR}_{\text{th}}\) at clipping probabilities of \(\text{Pr}_c = 10^{-4}, 10^{-6}\), and \(10^{-8}\) over \(U\) and \(N_t\). These results base on the analytic expression (9) of the ccdf. The trajectory of steepest descent (starting from \((U, N_t) = (1, 1)\)) is depicted in black.

Considering the original SLM approach (i.e., \(N_t = 1\)) and already high values of \(U\) it is difficult to obtain additional performance gains by increasing \(U\). This can be circumvented by applying dSLM. Hence, enlarging the number of assessed candidates \(U\) should come along with an appropriate choice of \(N_t\).

For smaller clipping probabilities \(\text{Pr}_c\), the slope of the trajectory gets steeper. Hence, to achieve lower clipping probabilities the parameter \(N_t\) gets more important. However, as the slope is always lower than 45\(^\circ\) degree, the number of alternative signal representations \(U\) is the more significant parameter.

The tables below the contour plots give a hint how to save computational complexity compared to original SLM (\(N_t = 1\)) by spending delay and using the dSLM approach assuming that both methods achieve the same performance i.e., clipping probability. For instance, at a clipping probability of \(\text{Pr}_c = 10^{-4}\) it is possible to either use SLM with \(U = 64\) candidates or dSLM with only \(U = 16\) candidates (reduced complexity) and a hyper frame length of \(N_t = 11\) (increased delay).

Fig. 4 shows the effect by introducing an additional delay \(\Delta_{\text{add}}\). The contour plot is depicted for a fixed number of candidates \((U = 4\) and \(8\) are assessed) and various hyper frame lengths \(N_t\); the clipping probability is \(\text{Pr}_c = 10^{-4}\). The lines of equal total delay are dotted.

For a small hyper frame length \(N_t\), an additional delay \(\Delta_{\text{add}}\) does not offer any advantages. If \(N_t\) is increased,
there is no big difference between applying a larger hyper frame length or an additional delay \( \Delta_{add} \). As computational complexity and delay are the same in both cases, it is more advisable to work on larger hyper frames.

Throughout this paper, we concentrated solely on the ccdf as performance measure. In [13], out-of-band radiation was examined by evaluation of the power spectral density of the transmit signal for MIMO dSLM. A similar reduction of the out-of-band radiation can be observed for the SISO dSLM approach.

**REFERENCES**


