Tone Reservation for Peak-to-Average Power Ratio Reduction in OFDM under Different Optimization Constraints

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Abstract—The high peak-to-average power ratio (PAR) of the transmit signal is considered to be one major drawback of orthogonal frequency-division multiplexing (OFDM) systems. This paper analyzes the tone reservation (TR) technique for PAR reduction. It is shown that the good performance of TR is paid with an increase in average transmit power. Therefore, an extension of the TR scheme is considered, which limits the average power. This increase of power efficiency comes along with a decrease of PAR reduction; a trade-off is possible. In addition to that, the TR scheme is compared with selected mapping (SLM), another popular PAR reduction technique. Simulations show that SLM outperforms TR in terms of PAR reduction and power efficiency. Finally, a combined approach of TR and SLM is proposed. Although, this scheme does not offer the PAR reduction capabilities of pure TR it has lower complexity.

I. INTRODUCTION

One of the most serious drawbacks of orthogonal frequency-division multiplexing (OFDM) [1] is the high peak-to-average power ratio (PAR) of the time-domain transmit signal. Non-linear power amplifiers at the transmitter front-end cause signal clipping, which, in turn, leads to signal distortion and out-of-band radiation. In order to avoid violation of spectral masks by out-of-band radiation, operation at large input power back-off is required. To decrease the input power back-off and therefore increase power efficiency, a transmitter-sided preprocessing to reduce the transmit signal’s peak-power is indispensable.

Over the last years, several strategies were discussed in literature to overcome this issue. One very popular technique is tone reservation (TR) [2]. With TR the minimization of the PAR of the transmit signal can be formulated as a convex optimization problem [3], which can be solved efficiently.

Main idea of this scheme is to reserve a number of subcarriers to produce a redundant, additive signal which reduces the peak-power. As TR works on reserved subcarriers (tones) no additional signal processing is required at the receiver. It only has to be aware of the positions of the reserved subcarriers and ignores these signals. Explicit transmission of side-information is not necessary in this case. This is done inherently as the reserved subcarriers are not available for transmitting information. Due to these reasons, tone reservation is quite popular for practical implementations and therefore is specified in 3GPP [4] as one possible PAR reduction technique, for instance.

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This paper gives a short review of the TR technique and addresses the problem of an increased transmit power. Moreover, the original TR algorithm is extended in order to limit the maximum possible increase in transmit power. In addition to that, these techniques are compared with selected mapping (SLM) [5], another popular technique for reducing the peak-power. Finally, a combination of TR and SLM is derived, which selects the best suited additive signal on the reserved tones among several given ones.

The paper is organized as follows: Section II defines the system model. The ideas and results of tone reservation are discussed in Section III, whereby a novel PAR reduction technique based on selection among different additive signals is introduced in Section IV. Finally, Section V draws some conclusions.

II. SYSTEM MODEL

Subsequently, an OFDM system with D subcarriers is considered. The frequency-domain OFDM symbol is denoted by the vector\(^A = [A_d], d = 0, \ldots, D-1\), of length \(D\). Each element \(A_d\) is i.i.d. drawn from an arbitrary \(M\)-ary QAM constellation. The frequency-domain OFDM symbol is transformed via the inverse discrete Fourier transform (IDFT) into the time-domain OFDM symbol \(a = \text{IDFT}(A)\) with \(a = [a_k], k = 0, \ldots, D-1\) and \(a_k = 1/\sqrt{D} \sum_{d=0}^{D-1} A_d \exp(2\pi/kd/D)\). The numerical results assume a 4-QAM modulation transmitted with OFDM using \(D = 512\) subcarriers.

Due to the superposition of \(D\) carriers, the OFDM time-domain signal exhibits a high peak-to-average power ratio

\[
\text{PAR} = \frac{\|a\|_\infty^2}{\|a\|_2^2}.
\]

This PAR definition is based on the, so-called, short-term average power, i.e., the average power over the actual considered OFDM frame.

III. TONE RESERVATION FOR PAR REDUCTION

In order to reduce the peak-power of the transmit signal, common PAR reduction techniques either introduce (i) distortion of the information carrying signal, (ii) redundancy which leads to rate-loss, and/or (iii) an increase of average transmit power.

\(^1\)Notation: Throughout this paper all vectors are assumed to be row vectors and are denoted as bold letters. Vectors in the frequency domain are written as capital letters, whereby vectors in time domain as lower case letters. The \(L\)-norm of a vector is written as \(\| \cdot \|_L; (\cdot)^T\) gives the transpose.
A. Idea of Tone Reservation

The idea of tone reservation is to define two sets, \( I_+ \) and \( I_- \), of \( D_+ \) active and \( D_- \) reserved subcarriers, respectively, with \( D_+ + D_- = D \). Both index sets are predetermined and common to transmitter and receiver and must fulfill the following conditions:

\[
I_+ \cup I_- = \{1, \ldots, D\} \quad \text{and} \quad I_+ \cap I_- = \emptyset. \tag{2}
\]

The information carrying signal is only transmitted over the active subcarriers, i.e., \( A_d = 0 \) for \( d \in I_- \). The signal of the reserved subcarriers \( C = [C_d] \) with \( C_d = 0 \) for \( d \in I_- \) can be chosen arbitrarily. The time-domain OFDM frame is now given by

\[
x = a + c = \text{IDFT}\{A + C\}. \tag{3}
\]

In time domain, the additive signal \( c \) influences the whole information carrying signal \( a \). Hence, the aim is to choose the frequency-domain additive signal \( C \) such that it minimizes the PAR of \( x \).

A block diagram of this PAR reduction scheme is depicted in Fig. 1.

![Block diagram of PAR reduction through additive signal](image)

As \( I_+ \) and \( I_- \) are disjoint sets, after discrete Fourier transform at the receiver it is possible to distinguish between information carrying signal \( A \) and additive signal \( c \). Hence, no signal distortion occurs.

However, as a number \( D_- \) of subcarriers is not used for transmission of information it can be regarded as introduced redundancy. Noteworthy, the additive signal \( c \) can arbitrarily chosen, i.e., the norm \( |c|^2 \) is not restricted and, moreover, an increase of average transmit power occurs.

In order to compare the impact of these issues on different setups of the PAR reduction system we regard the long-term power efficiency

\[
\eta_{\text{TR}} = \frac{\sigma_a^2 D_+}{\sigma_c^2 D} = \frac{\sigma_a^2 D_+}{\sigma_a^2 D_+ + \sigma_c^2 D_-} = \frac{1}{1 + \frac{\sigma_c^2}{\sigma_a^2} \frac{D_-}{D_+}}. \tag{4}
\]

the fraction between (long-term) average power \( \sigma_a^2 = \text{E}\{|a|^2\} \) of the information carrying signal \( a \) and the average power \( \sigma_c^2 = \text{E}\{|c|^2\} \) of the transmit signal \( x \). The long-term average power of the additive signal is given by \( \sigma_c^2 = \text{E}\{|c|^2\} \).

The peak-to-average power ratio of a given OFDM frame after applying the TR technique is defined by

\[
\text{PAR} = \frac{|x|^2_\infty}{|x|^2_2}.
\]

In this definition the power of the additive signal \( c \) is considered. Definition (5) is reasonable, as we are interested in the PAR of the actual transmitted signal \( x \).

Other definitions in literature dealing with the TR technique, do not incorporate the additive signal in order to take solely the peak-reducing characteristics of TR into account and may rather be characterized as peak-power reduction schemes.

B. Review of the TR Convex Optimization Problem

Subsequently, we address the problem to find a suitable additive signal \( c \). According to [2], the PAR reduction problem can be formulated as a convex optimization problem:

\[
\min p \\
\text{subject to} \quad (a_k + q_k^T C)(a_k + q_k^T C)^* \leq p \quad \forall k, \tag{6}
\]

whereby the vector \( q_k^T \) of length \( D \) represents the \( k \)th row of the inverse Fourier matrix [6], i.e., \( q_k = [q_{k,d}] \) with \( q_{k,d} = 1/\sqrt{D} \exp(j2\pi kd/D) \).

Optimization problems of this type are extensively discussed in literature. According to [3] a solution can, e.g., be found by application of interior-point methods.

Unfortunately, finding an optimum solution of the additive signal \( c \) is quite computational exhaustive. Although low complexity implementations do exist, e.g., [2], [7], [8], the complexity is one serious drawback of TR. In addition to that, the complexity is not fixed for each OFDM frame but may vary for different information carrying signals \( a \).

As the additive signal \( C \) has to be calculated for every OFDM frame individually, the (short-term) average power of one transmitted OFDM frame varies strongly. This fact helps to reduce the PAR, but also leads to a loss in power efficiency (4).

C. TR with Limited Average Transmit Power

In order to increase the power efficiency, the optimization problem from (6) can be extended by an additional constraint. We define a factor \( \xi \) by which the (short-term) average transmit power of one OFDM frame might be maximal exceeded, i.e., \( \xi = \frac{|x|^2_\infty}{\sigma_a^2} - 1 \). The corresponding optimization problem is now given by

\[
\min p \\
\text{subject to} \quad (a_k + q_k^T C)(a_k + q_k^T C)^* \leq p \quad \forall k \quad C^\dagger C \leq D_- \sigma_c^2 (1 + \xi \frac{D_-}{D_+}). \tag{7}
\]

A solution can also be found according to [3].

D. Numerical Results and Discussion

1) Tone Reservation: The top plot of Fig. 2 shows numerical results of the complementary cumulative distribution function (ccdf) of the tone reservation technique for various number \( D_- \) of used subcarriers. The respective numbers are \( D_- = 486 \approx 95\% \), \( D_- = 507 \approx 99\% \), \( D_- = 510 \), and \( D_- = 511 \). Evidently, the higher the number of reserved tones, the better the PAR reduction.

However, these significant gains, achieved through the tone reservation technique suffer from several drawbacks. As the additive signal \( c \) does not carry any information
it increases the average transmit power. In addition to that, the reserved carriers are not available for transmission of information. With a given modulation alphabet, these lost carriers can be interpreted as equivalent redundancy bits. The three bottom plots of Fig. 2 show the long-term fractional power increase $\frac{\sigma^2_2}{\sigma^2_0} - 1$ (in %) (top), the long-term power efficiency $\eta_{TR}$ (middle), and the number of equivalently used redundancy bits given by $\log_2(M) \cdot D_-$ (bottom) for various numbers $D_+$ of active subcarriers.

Transmitting data over $D_+ = 511$ subcarriers, i.e., using $D_- = 1$ subcarrier for PAR reduction does already lead to a significant gain in PAR distribution. However, the transmit power increases on average by approx. 4.5% whereby the power efficiency $\eta_{TR}$ decreases to approx. 95%; the equivalent number of redundancy bits is two bits per OFDM frame. Using even less subcarriers for transmission ($D_+ = 510$ or $D_+ = 507$) leads to higher gains in PAR reduction, but also leads to an increased average transmit power (up to more than 10% or less than 90% in power efficiency) and redundancy bits (up to 10 bits per OFDM frame).

Interestingly, for a (rather) small number of used subcarriers $D_+ = 486$ (i.e., $D_- = 26$) where the peak-power reduction performance is extraordinary good, the fractional power increase is even less than for $D_+ = 507$. In this case, the additional signal offers so many degrees of freedom that less power is sufficient to achieve very good performance. However, the number of equivalently required redundancy bits is extremely large (more than 50 bits) and not tolerable.

Fig. 3 shows the results of TR if the maximum increase of transmit power of the OFDM frames is limited to $\xi$. Here, we consider only $D_+ = 507$ and four values for $\xi = 0.12, 0.1, 0.5,$ and $0.01$. Evidently, the higher the allowed power increase, the better is the PAR reduction and the results converge to the ones of pure TR.

According to the formulation of the optimization problem, the parameter $\xi$ defines a limit on the short-term average power. An optimum solution for $c$ may not exploit this limit, but stay below. Hence, this limit is a worst-case assumption on the long-term average power $\sigma^2_2$. For reference, the limit $\xi$ is also depicted in the two bottom plots in Fig. 3.

Even if the loss in PAR performance is significant with this scheme, now a trade off between PAR reduction performance and power efficiency is present. The equivalent redundancy is not affected here and equal to pure TR and therefore not depicted.

Finally it can be stated, the PAR reduction performance is largely governed by an increase in transmit power. Only for a (comparatively) small number of used subcarriers the TR technique is able to reduce the peaks.

2) Comparison with Selected Mapping: In literature, selected mapping\(^2\) [5] is one popular technique for PAR reduction in OFDM systems. Advantages of SLM compared to TR are as follows. The “best suited” transmit signal is determined by straightforward calculations, i.e., $U$ calculations of IDFT and decision metric (PAR) are required. It is not necessary to design a complex algorithm in order to solve the optimization problem (6). Moreover,

\(^2\)The idea of SLM is to generate out of the original OFDM frame several, say $U$, different signal representations via $U$ different mappings. Out of these signal candidates, the best one, i.e., the one exhibiting the lowest PAR is chosen for transmission. At the receiver after equalization, the original data has to be reconstructed by inverting the applied mapping. Hence, side information, in terms of an index of the applied mapping, has to be transmitted.

A practical way to transmit the side information has been proposed in [9]. The so-called scrambler variant of SLM distributes the side information inherently over the whole OFDM frame and does not require its explicit transmission.
as SLM operates on different signal representations based on the same modulation scheme no explicit increase of the transmit power occurs.

In order to decode the signal correctly at the receiver, the mapping applied on the current OFDM frame has to be accomplished. Numerical simulations of the results of pure tone reservation are depicted in Fig. 3. Numerical results of tone reservation with limited transmit power transmitted. This redundancy is given by \( \left\lceil \frac{\log_2(M)}{2} \right\rceil \). This calculation is done in an optimum way by solving the convex optimization problem (6) or (7), respectively. The additive signal \( c \) has two effects on the transmit signal: on the one hand it decreases the peaks but on the other hand the average transmit power is increased. However, both effects are beneficial on the PAR but an increased average power lowers the power efficiency (4).

Assuming the transmission system tolerates some loss on the power efficiency, it is possible to design a novel PAR reduction technique which can be regarded as a combination of the ideas of selected mapping and tone reservation.

Instead of applying different mappings on the original OFDM frame it is also possible to create the number of candidates to complexity of only a variant of SLM is proposed, which reduces the performance of assessing \( U \) candidates to complexity of only \( 2\sqrt{U} \). Hence, the performance of SLM with \( U = 1024 \) is achievable by having the complexity of \( U = 64 \), which is tolerable. Nevertheless, \( U = 4.5 \cdot 10^{15} \) is a rather impractical setup. Due to this reason, the given ccdf curves are analytic results based on Gaussian signalling [5]. The results of Fig. 2 show, that TR is outperformed by SLM with respect to PAR reduction and power efficiency.

Hence, the peak-power reducing capabilities of SLM are more significant than of tone reservation as its good performance is primarily based on increasing the average transmit power.

IV. PAR Reduction through Selected Tone Reservation

The basic idea of the tone reservation technique is to calculate the additive time-domain signal \( c \) which reduces the PAR of the actual transmit signal \( x \). This calculation is done in an optimum way by solving the convex optimization problem (6) or (7), respectively. The additive signal \( c \) has two effects on the transmit signal: on the one hand it decreases the peaks but on the other hand the average transmit power is increased. However, both effects are beneficial on the PAR but an increased average power lowers the power efficiency (4).

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absolute value (in the range starting from \(|\gamma| = 0\) linearly increasing to \(|\gamma| = \sqrt{\xi D/D_0 + 1}\), whereby \(\xi\) represents again the worst-case power increase per OFDM frame). Noteworthy, the total number of assessed candidates is given by the product of \(U\) with the number of assessed values \(\gamma\).

Now, the results of numerical simulations showed a better performance in PAR reduction. However, it appears that almost all chosen values of \(\gamma\) exhibited the maximum possible absolute value whereby the phase is equally distributed.

**Approach III:** Hence, a more useful approach is to restrict the weighting factor’s absolute value to \(|\gamma| = \sqrt{\xi D/D_0 + 1}\). Here, only an optimization over the phase is required. Given an additive sequence \(e\), the resulting PAR values are non-monotonously distributed over the phase of \(\gamma\). Thus, finding the optimum value is quite exhaustive. We propose, to assess certain given values, here, e.g., \(\{0, \ldots, 7\} \cdot \pi/4\).

Fig. 4 shows numerical results of this technique compared to pure tone reservation. For each value of \(D_+\), the absolute value \(|\gamma|\) (or \(\xi\), respectively) is chosen that the increase of average power or power efficiency are equal to the ones of pure TR (cf. Fig. 2). Evidently, the performance increases if the number of candidates is increased.

Unfortunately, this approach is significantly outperformed by the pure tone reservation scheme, whereby for \(D_+ = 511\) and \(D_- = 510\) the loss is only approximately 1 dB. However, the benefit of this approach is the reduced complexity. With selected TR the complexity is mainly governed by the number of PAR evaluations (given by the product of \(U\) with the number of assessed values of \(\gamma\)). Hence, the complexity is fixed per OFDM frame. Our simulations of pure TR indicate that it requires much higher complexity than selected TR. However, a quantitative comparison remains to be done. Moreover, another drawback of pure TR is that the complexity per OFDM frame is not fixed but varying.

**V. Conclusions and Future Work**

This paper analyzes the tone reservation technique for PAR reduction in OFDM systems. Although TR offers quite good performance in PAR reduction, it suffers from several drawbacks. The most annoying point is the increase of transmit power which lowers the power efficiency.

Comparing TR with SLM in terms of equal (equivalent) redundancy, it is always outperformed by SLM, i.e., SLM offers better PAR reduction performance and better power efficiency. However, one unattractive point of SLM might be that receiver-side signal processing is necessary to invert the applied mapping. In contrast to that, TR is a pure transmitter-sided scheme.

The power inefficiency of TR might be avoided if an additional constraint, which limits the average power, is introduced in the optimization problem. As this method reduces PAR reduction performance a trade-off between both issues can be achieved.

**References**


