Signal Processing in Decision-Feedback Equalization of Intersymbol-Interference and Multiple-Input / Multiple-Output Channels: A Unified View

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Abstract

Digital transmission over dispersive channels which suffer from intersymbol interference (ISI) and/or over multiple-input/multiple-output (MIMO) channels which exhibit multi-user interference is reviewed. In particular, the digital signal processing required at the receiver for performing decision-feedback equalization (DFE) is studied. The aims of processing the received signal are discussed and compared for ISI channels, MIMO channels, and MIMO ISI channels. The calculation of the optimum filters for these applications is presented in a unified way, and it is shown that each aspect in one field has its immediate correspondence in the other application.

Key words: Decision-Feedback Equalization, Intersymbol-Interference Channel, Multiple-Input/Multiple-Output Channels

Dedicated to Prof. em. Dr.-Ing., Dr.-Ing. e.h., Dr. techn. e.h. Hans Wilhelm Schüßler on occasion of his 75th birthday

1 Introduction

Digital signal processing is the basis for all modern communication. Whenever new signal processing concepts are introduced and fast as well as low-cost implementation is facilitated, new communication systems are made possible or existing ones can be substantially improved.

In this paper, we review fast digital transmission over channels which suffer from intersymbol interference (ISI) and/or multi-user interference with emphasis on the digital signal processing required at the receiver. When transmitting over ISI channels, due to the dispersive nature of these channels, postcursors (echos) of previ-
ously sent data symbols disturb the current symbols and prevent its reliable detection. Similarly, when transmitting multiple data streams in parallel, e.g., using antenna arrays which create a multiple-input/multiple-output (MIMO) channel, the signals may severely interfere each other at the receiver side.

For achieving best performance, i.e., low error rates, these interferences have to be equalized. Here, we concentrate on decision-feedback equalization (DFE), where already detected symbols assist in the equalization. Applying the well-known chain rule of information theory (cf., e.g., [4]) to mutual information, it is straightforward to show that DFE is an optimum equalization strategy for ISI channels, in principle. For this purpose, discrete-time signal processing at the receiver has to shape the end-to-end impulse response, seen by the transmitted data, suitably. The aim of the present paper is to review these demands, and the strategies to calculate the filters required at the receiver. The basic concepts are enlightened and compared for ISI channels, MIMO channels, and their synthesis, MIMO ISI channels. Using a uniform presentation, it is shown that each approach known in one field parallels those for the other scenarios.

Note, throughout the paper we restrict ourselves to symbol-by-symbol detection schemes and do not consider maximum-likelihood sequence estimation [13] which is often much too complex to be implemented, see, e.g., [23, 12].

In Section 2 we consider transmission over ISI channels and review the properties of the optimum filters required for decision-feedback equalization. Section 3 shows the corresponding procedure for flat-fading, i.e., non-frequency-selective, MIMO channels, whereas in Section 4 the combination of both scenarios to MIMO channels with ISI, i.e., frequency-selective MIMO channels, is given. The paper closes with a brief summary (Section 5).

2 Equalization of Intersymbol Interference Channels

Consider the system model of pulse-amplitude-modulated (PAM) transmission over linear distorting channels, given in Fig. 1. If baseband signaling is studied, all signals and systems are real-valued, whereas for passband transmission signals and systems are complex-valued, given in the equivalent complex baseband domain, where we use the energy-conserving version of this transform, cf., e.g., [31].

The data are represented by means of a discrete-time sequence \( \langle a[k] \rangle \) of data symbols \( a[k] \), \( k \in \mathbb{Z} \), taken from the signal constellation \( A \). In order to obtain the continuous-time transmit signal \( s(t) \), this sequence of \( T \)-spaced symbols, where \( T \) denotes the duration of the modulation interval, is filtered by a continuous-time transmit filter with transfer function \( T \cdot H_T(\omega) \). The linearly distorting channel is described by its transfer function \( H_C(\omega) \), and the additive Gaussian noise \( n(t) \)
modeled to be present at the channel output, is assumed to be white with power spectral density $N_0$, i.e., $N_0/2$ per quadrature component.¹

The distorted and noisy receive signal $r(t)$ is passed through a receive filter $H_R(\omega)$, at the output of which $T$-spaced samples $y_o[k]$ are extracted. Based on these samples, the receiver has to produce estimates $\hat{a}[k]$ on the transmitted data symbols. Because both the transmitted data $a[k]$ and the sampled receive filter output $y_o[k]$ are $T$-spaced, in summary, a discrete-time end-to-end model of the transmission, including transmit filter, channel, and receive filter can be set up. The overall transfer function is denoted by $H_o(z)$, and an additive Gaussian noise sequence, $n_o[k]$, is present.

In this paper we discuss the choice of the receive filter, or equivalently, the demands for the induced discrete-time channel model, and the corresponding decision strategy. For this, a very general result due to T. Ericson [7] on the receive filter for PAM transmission is very useful: For any reasonable performance criterion, the optimal receive filter consists of the cascade of a matched filter for $H_T(\omega)H_C(\omega)$, $T$-spaced sampling, followed by a discrete-time filter.

Hence, as an intermediate step, we consider the matched-filter front-end $H_R(\omega) = H_T^*(\omega)H_C^*(\omega)$ and $T$-spaced sampling, to obtain a discrete-time transmission model with transfer function ($\Omega \equiv \omega T$)

$$H_o(e^{j\Omega}) = \sum_{l=-\infty}^{+\infty} H_T^*(\Omega+2\pi l/T)H_C^*(\Omega+2\pi l/T)H_C(\Omega+2\pi l/T)H_T(\Omega+2\pi l/T).$$ (1)

Noteworthy, for the present situation, the power spectral density of the additive noise $n_o[k]$ is proportional to the signal transfer function and reads

$$\Phi_{n_o,n_o}(e^{j\Omega}) = \sigma_n^2 H_o(e^{j\Omega}) \quad \text{with} \quad \sigma_n^2 \equiv \frac{N_0}{T}. \quad (2)$$

¹ If the noise is colored, a continuous-time noise whitening filter may be the first stage of the receive filter. Apportioning this filter conceptually to the channel, a modified channel transfer function and additive white noise result again. Thus, this model covers the very general situation.
Interestingly, the matched-filter frontend provides a lossless transition from the continuous-time to the discrete-time model. Even though the sampling theorem is usually not met, the samples at the matched filter output contain all information about the transmitted data symbols as long as the sampling phase is chosen appropriately [23].

2.1 Equalization Strategies

Now, we turn to the problem of how to choose the discrete-time part of the receive filter and how to generate decisions. At first glance, a natural choice would be to perform linear equalization. The transfer function, $H_o(z)$ (the analytic continuation of $H_o(e^{j\Omega})$, given on the unit circle, to the whole complex plane), experienced by the data is equalized by a filter $F(z) = 1/H_o(z)$. In this case the end-to-end impulse response reads $\delta[k]$ (unit pulse sequence), i.e., intersymbol interferences are no longer present and simple symbol-by-symbol threshold decisions can be performed to recover data. However, the additive noise is enhanced due to $1/H_o(z)$, and power efficiency of this linear equalization strategy usually is very poor. Moreover, in the case of spectral zeros of $H_o(z)$, linear equalization is even impossible.

A much more attractive equalization strategy, often used in practice, is decision-feedback equalization (DFE), shown in Fig. 2. First, the samples $y_o[k]$ are passed through a feedforward filter $F(z)$. This filter, however, does not primarily equalize the signal, but may even introduce additional interference. The overall signal transfer function now reads $H(z) \overset{df}{=} H_o(z)F(z)$ and the filtered noise, $n[k]$, has power spectral density $\Phi_{nn}(e^{j\Omega}) = \Phi_{nnn}(e^{j\Omega})|F(e^{j\Omega})|^2$.

Assuming that correct decisions $\hat{a}[k - \kappa]$, $\kappa = 1, 2, \ldots$, on the past data symbols

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Fig. 2. Decision-feedback equalization.
are already available, their contribution to the intersymbol interference can be subtracted out. This is done by the feedback filter $B(z) - 1$. Forcing the interference to be zero (zero-forcing (ZF) DFE; for the generalization to minimum mean-squared error (MMSE) DFE, see, e.g., [3, 12]) is achieved by selecting $B(z) = H(z)$. For optimum performance of this equalization strategy, the following demands have to be met:

- **Causality:** Since a successive processing of the receive sequence is to be performed, the discrete-time end-to-end response $H(z)$ has to be causal. Precursors of the sample at time index zero on which the decision is based could not be canceled in a causal manner. They would remain uncanceled and would contribute to the noise term. Moreover, without loss of generality, we assume proper scaling, such that the impulse response corresponding to $H(z)$ is monic, i.e., $H(z) = 1 + h[1] z^{-1} + h[2] z^{-2} + \cdots$.

- **Noise Whitening:** The noise at the output of the feedforward filter should be white, i.e., the noise samples should be uncorrelated. If correlations were present, the reliability of the decision could be improved by prediction of the current noise sample based on prior noise estimates, e.g., [23, 12]. This effect is equivalently achieved by a feedforward filter which acts as a noise whitening filter. Hence, for optimality $\Phi_{n_0 n_0}(e^{j\Omega}) |F(e^{j\Omega})|^2 = \text{const.}$ has to be demanded.

- **Minimum Phase Property:** The latter demand gives a constraint on the magnitude of the receive filter; the phase can still be chosen suitably. Since the decisions are based on the zeroth tap of the end-to-end impulse response and all other terms are canceled, this tap should be as large as possible. This is guaranteed if and only if $H(z)$ is minimum phase [27, 28, 25], where among all impulse responses with given squared magnitude spectrum the energy concentration in the first samples is maximized. Only if the end-to-end transfer function $H(z)$ is minimum phase and the filtered additive noise sequence is white, the maximum signal-to-noise ratio is achieved.

It should be noted that DFE suffers from decision errors which may propagate in the feedback loop. Moreover, immediate decisions are required for equalization. This prohibits a straightforward combination of DFE with channel coding, where decoding is done on (long) blocks of receive symbols. Both these drawbacks can be overcome by precoding [12], where, basically, the feedback loop of DFE is transferred to the transmitter. Since linear pre-equalization would boost transmit power and show the same poor performance as linear equalization at the receiver side, precoding techniques work nonlinearly using modulo arithmetics [30, 21]. Neglecting minor differences, precoding schemes perform identical to DFE. In particular, DFE/precoding in combination with powerful channel coding is optimum in the sense of information theory. Already in 1972, R. Price [26] has shown, that equalization by DFE/precoding in combination with channel coding schemes developed for the ISI-free additive white Gaussian noise channel achieves the capacity of the
underlying continuous-time channel. Due to this fact and because of its low complexity, DFE/precoding is commonly considered as a canonical structure for equalization [3].

2.2 Spectral Factorization

For the above-mentioned demands, the optimum discrete-time feedforward \( F(z) \) and feedback filter \( B(z) \) can be calculated for a given scenario. Having \( H_o(e^{j\Omega}) \) and its analytic continuation \( H_o(z) \), we write

\[
H_o(z) = S^*(z^{-*}) \cdot \sigma_s^2 \cdot S(z) ,
\]

where \( \sigma_s^2 \) is a scaling factor and \( S(z) = 1 + \sum_{k \geq 1} s[k]z^{-k} \) is required to be causal, monic, and minimum-phase. Then, \( S^*(z^{-*}) \), where \( z^{-*} \) abbreviates \( (z^*)^{-1} \), is anticausal, monic, and maximum-phase. The factorization (3) is possible if \( H_o(e^{j\Omega}) \) satisfies the Paley-Wiener condition [27, 25]

\[
\frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| \log \left( H_o(e^{j\Omega}) \right) \right| \, d\Omega < \infty ,
\]

and can preferably be performed in the cepstral domain. For details see, e.g., [24, 23, 14, 12].

Having performed the factorization, the requested feedforward filter is given by

\[
F(z) = \frac{1}{\sigma_s^2 S^*(z^{-*})} ,
\]

which results in the desired causal, monic, and minimum-phase end-to-end transfer function for the data symbols

\[
H(z) = F(z) \cdot H_o(z) = S(z) .
\]

In summary, the equivalent end-to-end transmission of the data symbols \( a[k] \) (with formal \( z \)-transform \( A(z) \)) to the input of the decision device reads

\[
Y(z) = H(z) A(z) + N(z) ,
\]

where \( N(z) \) is the formal \( z \)-transform of the noise sequence \( n[k] \). It immediately follows that this additive noise, present at the decision device, is white with spectral
density

$$\Phi_{nn}(z) = \Phi_{nn,0}(z) \cdot |F(z)|^2 = \sigma_n^2 \frac{1}{\sigma_z^2} \equiv \sigma_n^2 d^2,$$

(8)

where $$d^2 \equiv \frac{1}{\sigma_z^2}$$ has been used.

Since $$S(z)$$ is forced to be monic and all postcursors are cancelled the signal-to-noise ratio (SNR) at the input of the decision device calculates to ($$\sigma_n^2$$: variance of the PAM data symbols $$a[k]$$)

$$\text{SNR} = \frac{\sigma_a^2}{\sigma_n^2 d^2}.$$  

(9)

Finally we remark that the cascade of matched filter $$H^*_T(\omega)H^*_C(\omega)$$ and discrete-time feedforward filter $$F(z = e^{j\Omega})$$ can be combined into the overall receive filter, at the output of which $$T$$-spaced samples are generated. Because this filter (i) is the optimum receiver front-end for DFE/precoding, (ii) whitens the discrete-time noise sequence, and (iii) is matched to the transfer function seen by the data, it is called whitened matched filter (WMF) [13, 1].

Actually, the WMF is a so-called square-root Nyquist filter, e.g. [23]. It acts for limitation of noise bandwidth and phase equalization, but there is no equalization in magnitude. It should also be mentioned that separate implementation of the matched filter and the discrete-time feedforward filter $$F(z)$$ may cause high effort and demands on accuracy whereas an implementation of the combined continuous-time receive filter often can be approximated in an easy way. A favorite implementation may also be to use a quite arbitrary anti-aliasing filter together with fractionally spaced digital signal processing and subsequent decimation.

3 Transmission over Flat-Fading MIMO channels

Over the last years, transmission systems with multiple transmit and receive antennas have gained high attraction, since by this an extremely high spectral efficiency and hence data rates can be achieved, cf., e.g., [29, 16]. For brevity, here we restrict the discussion to systems with the same number of $$M$$ transmit and receive antennas. In this case, the spectral efficiency may be increased up to a factor of $$M$$ over that of a single-antenna scheme. The results are immediately valid for scenarios with a larger number of receive antennas than transmit antennas.

The basic concept of such systems is the transmission over so-called multiple-input/multiple-output (MIMO) channels. Assuming square-root Nyquist pulse shaping
for each data stream at the transmitter and the corresponding matched filter for each receive signal, a very compact discrete-time end-to-end model of the PAM transmission can be given in the equivalent complex baseband. Moreover, restricting ourselves for the moment to flat, i.e., non-dispersive, fading channels, successively transmitted symbols do not interfere and each time interval can be processed on its own.

Combining the symbols transmitted simultaneously in one symbol interval over the \( M \) antennas into the vector \( \mathbf{a} = [a_1, \ldots, a_M]^T \), and the corresponding receive samples into the vector \( \mathbf{y}_c = [y_{c,1}, \ldots, y_{c,M}]^T \), the input/output relation of the MIMO channel is given by

\[
\mathbf{y}_c = \mathbf{H}_c \mathbf{a} + \mathbf{n}_c.
\]  

(10)

The channel matrix \( \mathbf{H}_c = [h_{m\mu}] \) contains the fading gains \( h_{m\mu} \) from transmit antenna \( \mu \) to receive antenna \( m \), and the additive channel noise samples at each receive antenna are combined into the vector \( \mathbf{n}_c = [n_{c,1}, \ldots, n_{c,M}]^T \). It is usual to assume that the noise samples are uncorrelated and have the same variance, i.e., \( \mathbb{E}\{\mathbf{n}_c\mathbf{n}_c^H\} = \sigma_n^2 \mathbf{I} \) (\( \mathbf{I} \): identity matrix).

### 3.1 Equalization Strategies

The \( M \) data streams transmitted in parallel are superimposed at the receiver side and multiuser interference occurs. In order to enable reliable decisions on the data symbols, these interferences have to be equalized. This task is tightly related to the equalization of ISI single-input/single-output (SISO) channels, cf. [9, 11]. In contrast to temporal equalization of ISI channels, now spatial equalization for each time step has to be performed for recovering the parallel data streams.

Each equalization strategy known for ISI channels has its direct counterpart for MIMO channels, cf. [11]. Again, the most obvious equalization strategy is linear equalization. Here, a decision vector is generated by \( \mathbf{r} = \mathbf{H}_c^{-1}\mathbf{y}_c \), and the end-to-end description is given by the identity matrix. But, as for ISI channels, linear equalization suffers from noise enhancement, and hence may have poor power efficiency.

An interesting, meanwhile popular, detection algorithm for uncoded transmission over MIMO channels has been introduced in [19]. The main point of the algorithm is a pre-filtering followed by the successive detection of the data symbols \( a_m \) and interference cancellation. In the literature, this approach is known as Bell

\[ ^3 \mathbf{A}^T: \text{transpose of matrix } \mathbf{A}; \mathbf{A}^H: \text{Hermitian (i.e., conjugate) transpose}; \mathbf{A}^{-H}: \text{inverse of the Hermitian transpose of a square matrix } \mathbf{A}. \]
Laboratories Layered Space-Time Architecture (BLAST) [15]. Studying this algorithm, it turns out that it performs nothing else than a matrix DFE [10, 5, 18, 12], shown in Fig. 3.

Fig. 3. Matrix decision-feedback equalization for MIMO transmission.

Supposing $H_c$ is non-singular, without loss of optimality, the first stage of the receiver may consist of the “matched matrix” $H_c^H$. Then, the overall matrix seen by the data vector $a$ is $H_o \triangleq H_c^H H_c$, and the correlation matrix of the noise $n_o = H_c^H n_c$ reads $\sigma_n^2 H_c^H H_c$.

As for ISI channels, the task of the feedforward part (matrix $F$) is to shape the end-to-end response $H \triangleq FH_o$ and the correlation matrix $\sigma_n^2 F F^H$ of the additive noise to exhibit some desired properties. Successively, the data symbols are detected (estimates $\hat{a}_m$) and via the feedback part (matrix $B - I$) the interference caused by already detected symbols is canceled out.

The demands for the present case are analog to those for ISI channels:

- **Causality**: For a successive processing of the received symbols from index 1 through index $M$, spatial causality is required. That is, symbols $a_m$ may only interfere into the decision variable of symbols $a_\mu$, $\mu > m$. Then, data can be processed consecutively, neglecting yet undecided symbols. Mathematically, we hence require the end-to-end description $H = FH_o$ to be lower triangular. Moreover, without loss of generality, we assume proper scaling for a unit main diagonal ($h_{mm} = 1, m = 1, \ldots, M$).

- **Noise Whitening**: For optimum symbol-by-symbol decision, the noise samples at the output of the feedforward matrix have to be uncorrelated. As we have assumed that white channel noise is present, the entire feedforward processing has to consist of a unitary matrix $U$ followed by a diagonal scaling matrix $D$, i.e., $FH_c^H = DU$. Then, $E\{FH_c^H n_c n_c^H H_c F^H\} = D U \sigma_n^2 I U^H D^H = \sigma_n^2 DD^H = \sigma_n^2 \text{diag}(|d_1|^2, \ldots, |d_M|^2)$.

- **Detection Order**: The last degree of freedom, analog to achieving minimum-
phase property for transmission over ISI channels, is the ordering of detection of the symbols. Rarely, the natural ordering \( 1, 2, \ldots, M \) will achieve best performance; choosing the optimal detection order has a significant impact on the performance. The detection order can be incorporated into the channel matrix by virtually relabeling the transmit antennas, and hence permuting its columns. A permutation matrix \( P \) (non-singular matrix where each row and column contains a single one) can be introduced, which is chosen such that transmission over \( H_c P \) and detection in the natural ordering \( 1 \) through \( M \) gives the best performance [19, 32].

Noteworthy, in the same way that DFE can be replaced by precoding for ISI channels, this can be done for MIMO channels [12, 11]. Neglecting minor differences, this MIMO precoding performs as well as DFE, however, error propagation at the receiver is avoided, and channel coding schemes can directly be applied in the same way as for \( M \) parallel ideal additive white Gaussian noise (AWGN) or flat fading channels.

### 3.2 Factorization

Again, we have to calculate the optimum feedforward matrix \( F \) for a given scenario. The feedback matrix \( B \) is then immediately given by \( B = H \). For brevity, we assume the optimum detection ordering is already known, and included in the channel matrix \( H_c \) by suitably permuting its columns. Key point is a QR-type decomposition [20] of the channel matrix, i.e., writing \( H_c = U^H D^{-1} S \), where \( U \) is unitary, \( S \) is lower triangular with unit main diagonal, and \( D = \text{diag}(d_1, \ldots, d_M) \) is diagonal. With this we can immediately write \( H_o = H_c^H H_c \) in the form

\[
H_o = S^H \cdot \Sigma_s \cdot S ,
\]

where \( \Sigma_s = D^{-H} D^{-1} = \text{diag}(|d_1|^{-2}, \ldots, |d_M|^{-2}) \). This Cholesky factorization [20, 22] of \( H_o = H_c^H H_c \), replaces the spectral factorization (3) for ISI channels. The triangular, unit diagonal matrix \( S \) takes the place of the causal and monic polynomial \( S(z) \); the diagonal matrix \( \Sigma_s \) corresponds to \( \sigma_s^2 \). The factorization (11) is possible if \( H_o \) is positive definite, which is the case if \( H_c \) is non-singular [20, 22].

From \( S \) and \( \Sigma_s \), the feedforward matrix is given by

\[
F = \Sigma_s^{-1} \cdot S^{-H} ,
\]

which results in the desired causal and unit diagonal end-to-end description, since

\[
H = FH_o = \Sigma_s^{-1} S^{-H} S^H \cdot \Sigma_s \cdot S = S .
\]
Thus, the equivalent end-to-end transmission of the data symbols to the input of the decision device reads

\[ y = H a + n , \]  

(14)

where the additive noise vector \( n \) is white with correlation matrix

\[ \Phi_{nn} = \sigma_n^2 \cdot \text{diag}(|d_1|^2, \ldots, |d_M|^2) = \sigma_n^2 \Sigma_s^{-1} . \]  

(15)

Assuming correct previous decisions, the individual signal-to-noise ratios at the input of the decision devices are given by

\[ \text{SNR}_m = \frac{\sigma_a^2}{\sigma_n^2 |d_m|^2} , \quad m = 1, \ldots, M . \]  

(16)

Comparing these equations to (5) through (9), the analogy is evident. In particular (5) shows that after the matched matrix, “filtering” with \( S^{-H} \) is performed to achieve spatial causality, followed by scaling (for unit-gain transmission) by \( \Sigma_s^{-1} \). However, since \( \Sigma_s^{-1} S^{-H} H_s^H = \Sigma_s^{-1} S^{-H} S^H D^{-H} U = DU \), the entire feedforward part simply consist of a unitary matrix and a diagonal scaling matrix. This is in analogy to ISI channels, where in the case of white channel noise the feedforward part is an all-pass filter followed by scaling.

4 Equalization of MIMO channels with Intersymbol Interference

The principles explained above can straightforwardly be extended to MIMO channels with ISI. Here, joint spatial and temporal equalization has to be performed. As for flat-fading MIMO channels, a compact discrete-time equivalent complex baseband representation of the channel can be given. Assuming that the channel is constant during a transmission burst, the elements of the channel matrix will now be impulse responses, rather than constant gain factors.

Denoting the channel impulse response from transmit antenna \( \mu \) to receive antenna \( m \) by \( h_{C,m\mu}(t) \) the continuous-time receive signal at antenna \( m \) of the ISI MIMO channel reads (\(*\): convolution)

\[ y_m(t) = \sum_{\mu=1}^{M} \left( T \sum_{k=-\infty}^{+\infty} a_{\mu}[k] h_T(t - kT) \right) * h_{C,m\mu}(t) + n_m(t) \]

\[ = \sum_{\mu=1}^{M} \sum_{k=-\infty}^{+\infty} a_{\mu}[k] \cdot T \left[ h_T(\tau) * h_{C,m\mu}(\tau) \right]_{\tau=t-kT} + n_m(t) , \]  

(17)
where \( a_m[k] \) are the elements of the transmitted data vector \( a[k] \) at time index \( k \). The noise \( n_m(t) \) at the receive antennas (arranged into the vector \( n(t) \)) is assumed to be spatially and temporally white, i.e., \( \mathbb{E}\{n(t + \tau)n^H(t)\} = N_0 I \delta(\tau) \).

Analog to ISI SISO channels, the optimum receiver consists of a matrix of matched filters for \( H_C(\omega)H_T(\omega) \), where \( H_C(\omega) = [H_{C,m\mu}(\omega)] \) is the matrix of channel transfer functions, \( T \)-spaced sampling, followed by discrete-time processing [8, 9]. Applying the matrix matched filter the different propagation paths are aligned and the impulse energies are concentrated in the sampling instances \( kT \). This procedure again provides sufficient statistics, i.e., there is no loss of information with respect to data estimation although the sampling theorem is usually not satisfied.

The discrete-time matrix channel \( H_o(z) \), evaluated on the unit circle, is here given as

\[
H_o(e^{j\Omega}) = \sum_{l = -\infty}^{+\infty} H^*_T(\Omega + 2\pi l/T)H_C^\dagger(\Omega + 2\pi l/T)H_C(\Omega + 2\pi l/T)H_T(\Omega + 2\pi l/T),
\]

(18)

and the autocorrelation matrix of the additive noise \( n_o[k] \) reads \( \Phi_{n_o,n_o}(e^{j\Omega}) = \sigma_n^2 H_o(e^{j\Omega}) \), with \( \sigma_n^2 = N_0/T \).

### 4.1 Equalization Strategies

In the present situation, both intersymbol (temporal) and multiuser (spatial) interference occur, which have to be equalized in order to enable reliable decisions on the data symbols. As in the previous sections, we concentrate on (matrix) decision-feedback equalization and MIMO precoding, respectively. Fig. 3 still depicts the situation, however for combined spatial/temporal equalization matrix filters instead of constant matrices are present, in particular, the feedforward matrix filter \( F(z) \) and a corresponding feedback matrix filter \( B(z) = I \).

For a successive detection of the data symbols \( a_m[k] \) over space and time ("space-time processing"), the following demands have to be met:

- **Causality:** Here, spatial and temporal causality is required. The end-to-end description \( H(z) = \sum_{k \geq 0} H[k] z^{-k} \equiv F(z)H_o(z) \) has to have only terms for time indices greater or equal to zero. Having detected the data vectors \( a[\kappa] \), \( \kappa < k \), interference caused by them can be canceled in the feedback loop. Moreover, at each time instant \( k \), we want to detect the symbols \( a_m[k] \) successively. For that, symbol \( a_m[k] \) may only interfere into the decision variable of symbols \( a_{\mu}[k] \), \( \mu > m \). Hence, the zeroth coefficient of the end-to-end response, \( H[0] \), is forced to be lower triangular. Finally, as above, suited scaling of the parallel data streams at the receiver is assumed, such that \( H[0] \) has a unit main diagonal.
• **Noise Whitening:** Since we perform a symbol-by-symbol decision, the noise samples at the output of the feedforward matrix filter have to be spatially and temporally uncorrelated, i.e., its correlation matrix, evaluated on the unit circle, has to read
\[ F^H(e^{j\Omega}) \sigma_n^2 H_o(e^{j\Omega}) F(e^{j\Omega}) = \sigma_n^2 \text{diag}(|d_1|^2, \ldots, |d_M|^2). \]

• **Minimum Phase Property and Detection Order:** Since the decisions are based on the diagonal elements of the zeroth coefficient of the end-to-end impulse response and all other terms are canceled, these elements should be as large as possible. In [17] it is shown that the Frobenius norm [20] of \( H[0] \) is maximized if \( H(z) \) is minimum phase, which for matrices is defined as \( \det(H(z)) \) to have roots only inside the unit circle. Hence, for optimum performance, we require \( H(z) \) to be minimum phase. Moreover, as the last degree of freedom, the detection ordering of the symbols has to be selected suitably. Via a permutation matrix \( P \) we include this ordering in the channel matrix filter \( H_C(\omega) \) (permutation of the columns) and apply the natural ordering from 1 through \( M \).

### 4.2 Factorization

As in the previous cases, the optimum feedforward matrix filter \( F(z) \) (and hence the matrix filter \( B(z) = H(z) \)) can be calculated by performing a (spectral) factorization, which now reads (cf., e.g., [2, 12])
\[ H_o(z) = S^H(z^{-*}) \cdot \Sigma_s \cdot S(z). \] (19)

Here, \( \Sigma_s = \text{diag}(|d_1|^{-2}, \ldots, |d_M|^{-2}) \) is diagonal, and the matrix polynomial is expected to be causal, \( S(z) = \sum_{k \geq 0} S[k] z^{-k} \), and minimum-phase (\( \det(S(z)) \neq 0, |z| \geq 1 \)). Moreover, \( S[0] \) has to be lower triangular with unit main diagonal ("monic" matrix polynomial \( S(z) \)). With this additional constraint, the factorization (19) is unique [34].

Analog to the Paley-Wiener condition [25], cf. (4), the factorization (19) is possible if \( H_o(z) \) satisfies a condition [34], which can be written as
\[ \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| \log \left( \det \left( H_o(e^{j\Omega}) \right) \right) \right| d\Omega < \infty. \] (20)

Unfortunately, there is no easy-to-use factorization algorithm as in the scalar case, see, e.g., [33, 6, 34].

Given \( S(z) \) and \( \Sigma_s \), the feedforward matrix filter is obtained as
\[ F(z) = \Sigma_s^{-1} \cdot S^{-H}(z^{-*}), \] (21)
which leads to the requested causal, minimum-phase, and monic end-to-end transfer function

\[ H(z) = F(z)H_o(z) = \Sigma_s^{-1}S^{-H}(z^{-s})S^H(z^{-s})\Sigma_s \cdot S(z) = S(z) . \]  

(22)

The equivalent end-to-end transmission of the data symbols to the input of the decision device here reads

\[ Y(z) = H(z) A(z) + N(z) , \]  

(23)

where \( A(z) \), \( N(z) \), and \( Y(z) \) denote the formal \( z \)-transforms of the vector sequences \( a[k] \), \( n[k] \), and \( y[k] \), respectively. As in the flat-fading MIMO case, the additive noise \( n[k] \) is spatially/temporally white with correlation matrix

\[ \Phi_{nn}(z) = \sigma_n^2 \cdot \text{diag}(|d_1|^2, \ldots, |d_M|^2) = \sigma_n^2 \Sigma_s^{-1} . \]  

(24)

Neglecting error propagation, the individual signal-to-noise ratios at the input of the decision devices read

\[ \text{SNR}_m = \frac{\sigma_a^2}{\sigma_n^2 |d_m|^2} , \quad m = 1, \ldots, M . \]  

(25)

When comparing these equations to (5) through (9) for ISI channels and (12) through (16) in the case of flat-fading MIMO channels, the synthesis of both domains in the present case of ISI MIMO channels is apparent.

Noteworthy, the optimum receive filter (which consists of the continuous-time matrix matched filter and the entire feedforward processing), is known as *multiple whitened matched filter* and has first been described by W. Van Etten [9]. It (i) is the optimum receiver front-end for DFE/precoding and space/time maximum-likelihood sequence estimation, (ii) temporally and spatially whitens the discrete-time noise sequence, and (iii) is matched to the transfer functions experienced by the transmitted data.

In practice, usually only \( M \) parallel continuous-time square-root Nyquist compromise filters are used before \( T \)-spaced sampling, combined with discrete-time matrix signal processing. But it has to be mentioned that in this case the signals from one transmit antenna to all receive antennas cannot be aligned in phase before sampling. Therefore, such an approach is sub-optimum in principle. The favorite solution in practice is to apply parallel anti-aliasing filters for all \( M \) receive paths, fractionally spaced matrix signal processing, and subsequent decimation.
5 Summary and Conclusions

In this paper, optimum signal processing for transmission over ISI, MIMO, and ISI MIMO channels has been reviewed. Via pre-filtering of the received noisy and distorted symbols specific properties have to be guaranteed. In particular, for decision-feedback equalization or precoding, a causal, monic, and minimum-phase end-to-end impulse response is to be aspired. Moreover, the filtered channel noise, present at the decision device, has to be white with respect to space and time. In each case, given the underlying channel, the optimum filters/matrices are obtained by solving a factorization problem.

The similarities between the three studied transmission scenarios are enlightened and the correspondences are stated. Strategies known from temporal equalization of ISI channels can straightforwardly be transferred to spatial equalization of flat fading MIMO channels or to combined spatial/temporal equalization of ISI MIMO channels.

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